The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #3 Solution

For1)

when $z = 0$ $\cos\sqrt{x^2+y^2} = 0$ $\sqrt{x^2+y^2} = \frac{\pi}{2} + k\pi$,we can choose $k = 0$

so $D = \{(x,y); x^2+y^2 \le \left(\frac{\pi}{2}\right)^2\}$ a circular disk

and $V = \iint\limits_D \cos\sqrt{x^2+y^2}dxdy$(polar coord.)$=\iint\limits_{D^*} r\cos r\ drd\theta$

where $D^* = \{(r,\theta); 0 < r \le \frac{\pi}{2}; 0 \le \theta < 2\pi\}$ so by parts

$$V = 2\pi\int_0^{\frac{\pi}{2}} r\cos r\ dr = 2\pi\left[r\sin r - \int \sin r\ de\right]_0^{\frac{\pi}{2}} = \pi^2 + 2\pi\left[\cos r\right]_0^{\frac{\pi}{2}} = \pi^2 - 2\pi.$$

For 2)

using polar coord. $\iint\limits_D e^{3(x^2+y^2)}dxdy = \iint\limits_{D^*} e^{3r^2}rdrd\theta = \pi\left[\frac{e^{3r^2}}{6}\right]_1^2 = \frac{\pi}{6}\left(e^{12} - e^3\right)$

where $D^* = \{0 \le \theta \le \pi, 1 \le r \le 2\}$ region between two circles, only top half

For 3)

$T = \{0 \le x \le 2, 2x \le y \le 4\}$ so

$$I = \iint\limits_T \frac{1}{(y-2x)^k}\ dA = \int_0^2 \left(\int_{2x}^4 (y-2x)^{-k}\ dy\right)dx$$

recall $$\int_0^1 u^{-k}du = \begin{cases} k = 1 & [\ln u]_0^1 = \infty \\ k > 1 & \left[\frac{u^{1-k}}{1-k}\right] = \infty \\ k < 1 & \left[\frac{u^{1-k}}{1-k}\right] = \frac{1}{1-k} \end{cases}$$

similarly for the inside integral $\int_0^2 \left[\frac{(y-2x)^{1-k}}{1-k}\right]_{y\to 2x}^{y=4} dx$

for $k \ne 1$ and $\lim\limits_{y\to 2x}(y-2x)^{1-k} = 0$ and finite only for $1-k > 0$

for $k = 1$ $[\ln(y-2x)]_{y\to 2x}^{y=4} = +\infty$

so **for only $k < 1$ the integral is convergent** and

$$I = \int_0^2 \left[\frac{(4-2x)^{1-k}}{1-k}\right]dx = \left[\frac{(4-2x)^{2-k}}{(-2)(1-k)(2-k)}\right]_0^2 = \frac{2}{(1-k)(2-k)} \text{since } 2-k > 0$$

For 4)

the region is in the first quadrant inside the circle$r = \sqrt{2}$,on the right of vert line $x = 1$

intersection of the line $x = 1$ and the circle is at the point (1,1)$\to \theta = \frac{\pi}{4}$

use polar coord.: $x \ge 1 \to r\cos\theta \ge 1$

$D^* = \{0 \le r \le \sqrt{2}, \theta \in \left[0, \frac{\pi}{4}\right], r \ge \frac{1}{\cos\theta}\}$

necessary cond.from $\frac{1}{\cos\theta} \le r \le \sqrt{2}$ is $\cos\theta \ge \frac{1}{\sqrt{2}}$ so $\theta \in \left[0, \frac{\pi}{4}\right]$

$$\iint\limits_{D}\frac{dxdy}{\sqrt{x^2+y^2}}=\iint\limits_{D^*}\frac{rdrd\theta}{\sqrt{r^2}}=\int\limits_0^{\frac{\pi}{4}}\left(\int\limits_{\frac{1}{\cos\theta}}^{\sqrt{2}}dr\right)d\theta=\int\limits_0^{\frac{\pi}{4}}\left(\sqrt{2}-\frac{1}{\cos\theta}\right)d\theta=$$

$$(\text{ Table})=\frac{\pi\sqrt{2}}{4}-\left[\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\frac{\pi}{4}}=\frac{\pi\sqrt{2}}{4}-\left[\ln\left|\sqrt{2}+1\right|-\ln 1\right]=$$

$$=\frac{\pi\sqrt{2}}{4}-\ln\left(\sqrt{2}+1\right)$$

For 5)

using polar coord.: $y=x \quad\rightarrow\quad \theta=\frac{\pi}{4}$

$x,y\geq 0 \quad\rightarrow\quad \theta\leq\frac{\pi}{2}$

$$\iint\limits_{D}\frac{x\sin\pi\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}dxdy=\iint\limits_{D^*}\frac{r\cos\theta\sin\left(\pi r^2\right)}{r}\ rdrd\theta=$$

$$=\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\theta d\theta\cdot\int\limits_0^1 r\sin\pi r^2\ dr=\left[\sin\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cdot\left[\frac{-\cos\pi r^2}{2\pi}\right]_0^1=\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\pi}\right).$$

where $D^*=\{\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2},0<r\leq 1\}$

For 6)

$D=\{x\in(1,+\infty),0\leq y\leq\frac{1}{x^2}\}$ is unbounded

$$I=\iint\limits_{D}e^{-x^2y}dA=\int\limits_1^{\infty}\left(\int\limits_0^{\frac{1}{x^2}}e^{-x^2y}dy\right)dx=\int\limits_1^{\infty}\left[\frac{e^{-x^2y}}{-x^2}\right]_{y=0}^{y=\frac{1}{x^2}}dx=\int\limits_1^{\infty}\left[\frac{e^{-1}-1}{-x^2}\right]dx=$$

$$=\left(\frac{1}{e}-1\right)\left[\frac{1}{x}\right]_{x=1}^{x\to\infty}=1-\frac{1}{e}.$$

For 7a)

the region is inside the circle with $r=\sqrt{2}$ above the horizontal line $y=1$

the intersection of the circle and the line $y=1$ is at $x=\pm 1$

so $$\iint\limits_{D}(x^2+y^2)dxdy=\int\limits_{-1}^{1}\left(\int\limits_1^{\sqrt{2-x^2}}\left(x^2+y^2\right)dy\right)dx=$$

$$\int\limits_{-1}^{1}\left(x^2\sqrt{2-x^2}+\tfrac{1}{3}\left(2-x^2\right)^{\frac{3}{2}}-x^2-\tfrac{1}{3}\right)dydx=...$$

For b)

in polar coord.:$(1,1)\rightarrow\theta=\frac{\pi}{4}$ and $(-1,1)\rightarrow\theta=\frac{3}{4}\pi$

$$\iint\limits_{D}(x^2+y^2)dxdy=\iint\limits_{D^*}r^3drd\theta$$

where $D^*=\{0<r\leq\sqrt{2}\ ,\ r\geq\frac{1}{\sin\theta},\theta\in\left[\frac{\pi}{4},\frac{3}{4}\pi\right]\}$

since $y\geq 1$ implies $r\sin\theta\geq 1$ so $\sin\theta$ must be positive and $\sqrt{2}\geq\frac{1}{\sin\theta}$

$$=\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\int\limits_{\frac{1}{\sin\theta}}^{\sqrt{2}}r^3\ dr\right)d\theta=\frac{1}{4}\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(4-\csc^4\theta)d\theta=\frac{\pi}{2}-\frac{2}{4}\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(1+\cot^2\theta\right)\csc^2\theta d\theta=$$

using symmetry $(\pm x)$ and $1+\cot^2\theta=\csc^2\theta$ then

subst. $u=\cot\theta \qquad du=-\csc^2\theta d\theta$

$$=\frac{\pi}{2}-\frac{1}{2}\int\limits_0^1\left(1+u^2\right)du=\frac{\pi}{2}-\frac{2}{3}.$$

For 8)

$D = \{x \in (1, +\infty), 0 \le y \le x\}$ is unbounded

$$I = \iint_D e^{-x-y} dA = \int_1^\infty e^{-x}\left(\int_0^x e^{-y} dy\right)dx = \int_1^\infty e^{-x} \left[-e^{-y}\right]_{y=0}^{y=x} dx = \int_1^\infty \left[e^{-x} - e^{-2x}\right] dx =$$

$$= \left[-e^{-x} + \tfrac{1}{2}e^{-2x}\right]_{x=1}^{x\to\infty} = \frac{1}{e} - \frac{1}{2e^2} \qquad (\text{"}e^{-\infty}\text{"} = 0)$$

For 9a)

the region is inside the circle with $r = 1$ and above the line $x + y = 1$

in the first quadrant

intersection at $(1, 0)$ and $(0, 1)$

so $$\iint_D \frac{1}{(x^2+y^2)^2} dxdy = \int_0^1 \left(\int_{1-x}^{\sqrt{1-x^2}} \frac{1}{(x^2+y^2)^2} dy\right) dx = \text{difficult}$$

For b)

in polar coord.:$x + y \ge 1 \to r(\cos\theta + \sin\theta) \ge 1$

$$\iint_D \frac{1}{(x^2+y^2)^2} dxdy = \iint_{D^*} r^{-3} dr d\theta$$

where $D^* = \{0 < r \le 1 \, , \, r \ge \dfrac{1}{\cos\theta + \sin\theta}, \theta \in \left[0, \dfrac{\pi}{2}\right]\}$

$$= \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^{-3} \, dr\right) d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - (\cos\theta + \sin\theta)^2) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta =$$

(using $\sin 2\theta = 2\sin\theta\cos\theta$) $= \dfrac{1}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \dfrac{1}{2}.$

For 10)

the function is unbounded

the set is the region between both axes, line $y = 1$ and $x = e^y \Leftrightarrow y = \ln x$

$$I = \iint_D \frac{1 + \ln x}{y} dA = \int_0^1 \frac{1}{y}\left(\int_0^{e^y} (1 + \ln x)\, dx\right)dy = \int_0^1 \frac{1}{y} \left[x \ln x\right]_0^{e^y} dy = \int_0^1 \frac{1}{y}(e^y y)\, dy$$

$$= e - 1 \qquad \left(\lim_{x\to 0+} x \ln x = 0\right)$$

if we switch the order of integration we have to split the set into

$D_1 = \{0 \le x \le 1, 0 \le y \le 1\}$ and $D_2 = \{1 \le x \le e, \ln x \le y \le 1\}$

then

$$.I = \iint_D \frac{1 + \ln x}{y} dA = \int_0^1 \frac{1}{y} dy \cdot \int_0^1 (1 + \ln x)\, dx + \int_1^e (1 + \ln x) \left(\int_{\ln x}^1 \tfrac{1}{y} dy\right)$$

much harder..

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