The University of Calgary
Faculty of Science
Department of Mathematics and Statistics
MATH 353
Midterm Supplement

1. **for a)** $\rho = x \to \sqrt{x^2+y^2+z^2} = x$

so $x \geq 0$ and $x^2+y^2+z^2 = x^2$, $y^2+z^2 = 0$

thus $y = z = 0$ and any point $(x,0,0)$ for $x \geq 0$ - half of the x-axis

for b) $\rho = -2y \to \sqrt{x^2+y^2+z^2} = -2y$

so $y \leq 0$ and $x^2+y^2+z^2 = 4y^2$

$x^2+z^2 = 3y^2$,one part cone around the y - axis $y = -\dfrac{\sqrt{x^2+z^2}}{\sqrt{3}}$.

for a)

first $B = \{(x,y,z)\,;x\geq 0, y\geq 0, z\geq 0, \dfrac{3x+2y}{6} \leq z \leq 2\}$

"bottom" must be below "top" $\dfrac{3x+2y}{6} \leq 2$ and we get

$D = \{(x,y)\,;x\geq 0, y\geq 0, 3x+2y \leq 12\}$ a triangle

$$\text{so} \quad \iiint\limits_B z\ dxdydz = \iint\limits_D \left(\int_{\frac{3x+2y}{6}}^{2} z\ dz\right) dxdy = \int_0^4 \left[\int_0^{\frac{12-3x}{2}} \left(\int_{\frac{3x+2y}{6}}^{2} z\ dz\right) dy\right] dx = ..$$

for b)

$0 \leq z \leq 2$ and for a fixed z

$D_z = \{(x,y)\,;x\geq 0, y\geq 0, 3x+2y \leq 6z\}$

triangles with the base $[0,2z]$ and the height $[0,3z]$ and the area $3z^2$

$$\iiint\limits_B z\ dxdydz = \int_0^2 z\left(\iint\limits_{D_z} dxdy\right) dz = \int_0^2 z\,(\text{area of } D_z)\, dz = 3\int_0^2 z^3 dz = 12$$

OR

$$\iiint\limits_B z\ dxdydz = \int_0^2 z\left(\int_0^{2z}\Big(\int_0^{\frac{6z-3x}{2}} dy\Big)dx\right) dz = ...$$

$B = \{(x,y,z)\,;x^2+y^2+z^2 \leq 4\ ;x^2+y^2 \geq 3, x\geq 0, y \geq 0\}$

for a)

the solid B is outside the cylinder and inside the sphere

$x\geq 0, y\geq 0$ implies $\theta \in \left[0, \frac{\pi}{2}\right]$

$x^2+y^2+z^2 \leq 4 \to 0 \leq \rho \leq 2$

and $x^2+y^2 \geq 3$ implies $\rho\sin\phi \geq \sqrt{3}$ $\quad \dfrac{\sqrt{3}}{\sin\phi} \leq \rho \leq 2$

so necessarily $\frac{\sqrt{3}}{2} \leq \sin\phi$ $\quad \phi \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$

$B^* = \{ \ \theta \in \left[0, \frac{\pi}{2}\right], \phi \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right], \dfrac{\sqrt{3}}{\sin\phi} \le \rho \le 2\}$

as iterated integrals

$$\iiint\limits_B \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \iiint\limits_{B^*} \frac{\rho^2 \sin\phi d\rho d\phi d\theta}{\rho} = \frac{\pi}{2}\int\limits_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \left(\int\limits_{\frac{\sqrt{3}}{\sin\phi}}^{2} \rho \ d\rho \right) d\phi =$$

$$= \frac{\pi}{4}\int\limits_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \left(4 - \tfrac{3}{\sin^2\phi}\right) d\phi = \pi\left[-\cos\phi\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} - \frac{3\pi}{4}\left[\ln\left|\csc\phi - \cot\phi\right|\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} =$$

$\left(\cos\frac{\pi}{3} = \frac{1}{2}, \cos\frac{2\pi}{3} = -\frac{1}{2}, \sin\left(\text{both}\right) = \frac{\sqrt{3}}{2}, \cot = \pm\frac{1}{\sqrt{3}}\right)$

$= \pi - \frac{3\pi}{2}\ln\sqrt{3} = \pi\left(1 - \frac{3}{4}\ln 3\right)$

also from the symmetry $I = 2 \cdot (\text{integral for } z > 0) \to 2 \cdot \{\phi \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]\}$

for b)

the solid B is outside the cylinder and inside the sphere

$x \ge 0, y \ge 0$ implies $\theta \in \left[0, \frac{\pi}{2}\right], x^2+y^2+z^2 \le 4 \to r^2+z^2 \le 4$

and $x^2+y^2 \ge 3$ implies $\quad r \ge \sqrt{3}$

$B^* = \{ \ \theta \in \left[0, \frac{\pi}{2}\right], r \ge \sqrt{3}, r^2+z^2 \le 4\}$

$$\iiint\limits_B \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \iiint\limits_{B^*} \frac{rdrdzd\theta}{\sqrt{r^2+z^2}} = \frac{\pi}{2}\int\limits_{\sqrt{3}}^{2} r \left(\int\limits_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \frac{dz}{\sqrt{r^2+z^2}} \right) dr =$$

$$\text{OR} = \frac{\pi}{2}\int\limits_{-1}^{1} \left(\int\limits_{\sqrt{3}}^{\sqrt{4-z^2}} \frac{r \ dr}{\sqrt{r^2+z^2}} \right) dz = \pi\int\limits_0^1 \left(\sqrt{r^2+z^2}\right)_{\sqrt{3}}^{\sqrt{4-z^2}} dz = \pi\int\limits_0^1 \left(2 - \sqrt{3+z^2}\right) dz =$$

$$= 2\pi - \tfrac{\pi}{2}\left[z\sqrt{3+z^2} + 3\ln\left|z + \sqrt{3+z^2}\right|\right]_0^1 = 2\pi - \tfrac{\pi}{2}\left[2 + 3\ln 3 - 3\ln\sqrt{3}\right] =$$

$= \pi\left[1 - \frac{3}{4}\ln 3\right].$

the solid is a tetrahedron in the first octant under the plane $3x+3y+z=6$

so $\qquad B = \{x \ge 0, y \ge 0, z \ge 0 \ , 3x+3y+z \le 6\}$

for $z = 0$ we have a triangle $(0,0), (2,0), (0,2)$ also D_0

for $z = 6 \qquad\qquad x = y = 0 \qquad$ point $(0,0,6)$

for $z \in (0,6) \qquad$ we have a smaller triangle

under the line $3x+3y = 6-z$ with vertices $(0,0), \left(0, \frac{6-z}{3}\right), \left(\frac{6-z}{3}, 0\right)$

$D_z = \{x \ge 0, y \ge 0, x+y \le \frac{6-z}{3}\}$ ...triangle with area $A_z = \frac{1}{2}\left(\dfrac{6-z}{3}\right)^2$

$$\iiint\limits_B \frac{dxdydz}{z-6} = \int\limits_0^6 \frac{1}{z-6}\left(\iint\limits_{D_z} dxdy\right) dz = \int\limits_0^6 \frac{1}{z-6} \cdot A_z \ dz = \frac{1}{18}\int\limits_0^6 (z-6) \ dz =$$

$= \frac{1}{36}\left[(z-6)^2\right]_0^6 = \frac{-36}{36} = -1.$

2. $B = \{x^2 + y^2 + z^2 \leq 2, z \geq 0, y \geq 0, x^2 + y^2 = z\}$

for a)

the set is inside the sphere with radius $\sqrt{2}$

below the paraboloid $z = x^2 + y^2$, and above the xy-plane $z = 0$

$B^* = \left\{r^2 + z^2 \leq 2, z \geq 0, r \geq 0, \theta \in [0, \pi], r^2 \geq z\right\}$

so $I = \iiint\limits_B z\ dxdydz = \iiint\limits_{B^*} z\ rdrd\theta dz = \pi \iint\limits_D rz\ drdz$

where $D = \left\{r^2 + z^2 \leq 2, z \geq 0, r \geq 0, r^2 \geq z\right\}$

the intersection of the circle and parabola is at $z = r = 1$

it is easier to slice it horizontally in $rz-$ plane

so $\quad z \in [0, 1]$ and $r \geq \sqrt{z}$, $r \leq \sqrt{2 - z^2}$

the integral $I = \pi \int\limits_0^1 z \left(\int\limits_{\sqrt{z}}^{\sqrt{2-z^2}} rdr \right) dz = \pi \int\limits_0^1 z \left(\left[\frac{r^2}{2}\right]_{\sqrt{z}}^{\sqrt{2-z^2}} \right) dz =$

$= \frac{1}{2}\pi \int\limits_0^1 z \left(\left[2 - z^2 - z\right]\right) dz = \frac{\pi}{2} \left[z^2 - \frac{z^4}{4} - \frac{z^3}{3}\right]_0^1 = \frac{\pi}{2} \cdot \left(1 - \frac{7}{12}\right) = \frac{5}{24}\pi.$

for b)

$x^2 + y^2 + z^2 \leq 2 \rightarrow \rho \leq \sqrt{2}$

$z \geq 0 \rightarrow \phi \in \left[0, \frac{\pi}{2}\right] \qquad y \geq 0 \rightarrow \theta \in [0, \pi]$

$x^2 + y^2 \geq z \rightarrow \rho^2 \sin^2 \phi \geq \rho \cos \phi \rightarrow \rho \geq \dfrac{\cos \phi}{\sin^2 \phi}$

but necessarily $\quad \sqrt{2} \geq \dfrac{\cos \phi}{\sin^2 \phi} \qquad \rightarrow \phi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

thus

$B^* = \left\{\rho \leq \sqrt{2}, \theta \in [0, \pi], \phi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \rho \geq \dfrac{\cos \phi}{\sin^2 \phi}\right\}$

and

$I = \iiint\limits_B z\ dxdydz = \iiint\limits_{B^*} \rho^3 \cos \phi \sin \phi\ d\rho d\phi d\theta =$

$= \int\limits_0^\pi d\theta \cdot \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cos \phi \sin \phi \int\limits_{\frac{\cos \phi}{\sin^2 \phi}}^{\sqrt{2}} \rho^3 d\rho \right) d\phi ...$

$B = \left\{x^2 + y^2 \leq z \leq 4 - x^2 - 3y^2\right\}$ between two paraboloids

necessarily $x^2 + y^2 \leq 4 - x^2 - 3y^2$

so $2x^2 + 4y^2 \leq 4$ that is $(x, y) \in D_0 = \left\{x^2 + 2y^2 \leq 2\right\}$ $-$ellipse

and

$$\iiint\limits_B \sqrt{x^2+2y^2}dxdydz = \iint\limits_{D_0} \sqrt{x^2+2y^2}\left(\int\limits_{x^2+y^2}^{4-x^2-3y^2} dz\right)dxdy =$$

$$= \iint\limits_{D_0} \sqrt{x^2+2y^2}\left(4-2\left(x^2+2y^2\right)\right)dxdy$$

use modified polar coord. to evaluate

$$x = r\cos\theta, y = \frac{1}{\sqrt{2}}r\sin\theta$$

then $\qquad x^2+2y^2 = r^2 \qquad dxdy = \frac{1}{\sqrt{2}}rdrd\theta$

so D_0 transforms into $D^* = \left\{0 \le r \le \sqrt{2}, 0 \le \theta \le 2\pi\right\}$

and the integral into

$$= \int\limits_0^{2\pi} d\theta \int\limits_0^{\sqrt{2}} \left(4r-2r^3\right)\frac{r}{\sqrt{2}}dr = \sqrt{2}\pi\left[\frac{4r^3}{3}-\frac{2r^5}{5}\right]_0^{\sqrt{2}} = \frac{32}{15}\pi.$$

$B = \left\{x \ge 0, y \ge 0, z \ge 0, z \le 2-y, x \le 4-y^2 \left(\text{ or } y \le \sqrt{4-x}\right)\right\}$

so

$D_0 = \left\{x \ge 0, y \ge 0, x \le 4-y^2 \left(\text{ or } y \le \sqrt{4-x}\right)\right\}$

and for a)

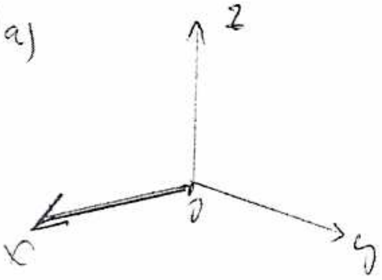
$$\iiint\limits_B f dxdydz = \iint\limits_{D_o}\left(\int_0^{2-y} f dz\right)dxdy$$

for b)

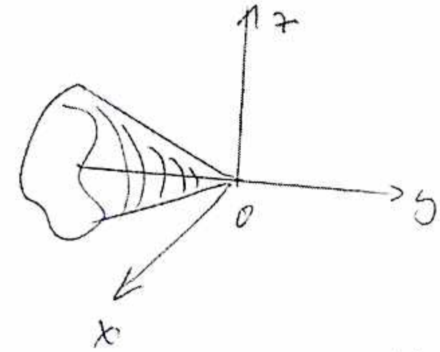
$0 \le z \le 2$ and for a fixed $z \qquad D_z = \left\{0 \le y \le 2-z, 0 \le x \le 4-y^2\right\}$

$$\iiint\limits_B f dxdydz = \int\limits_0^2\left(\iint\limits_{D_z} f dxdy\right)dz$$

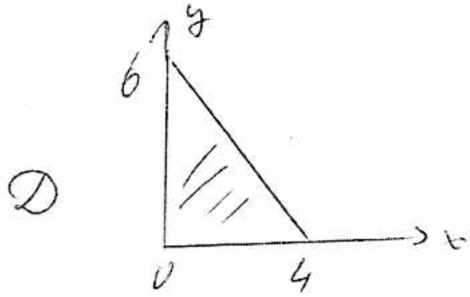
1a)



1b)



2a)



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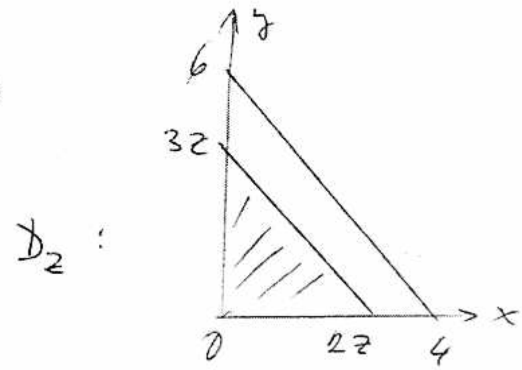
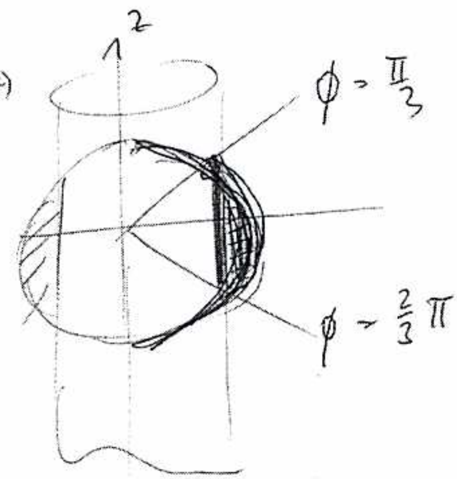
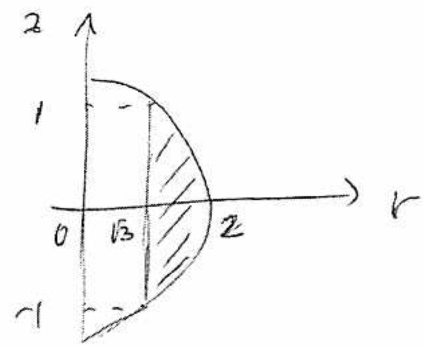


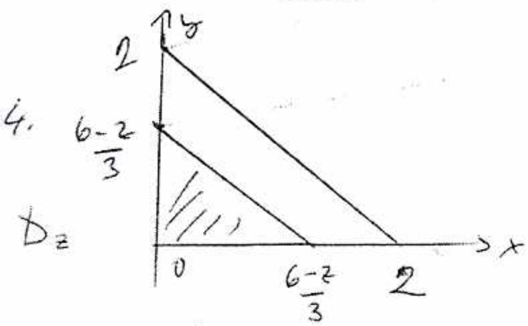
3. a)



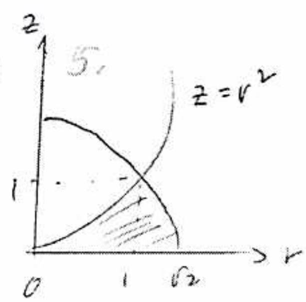
3b)



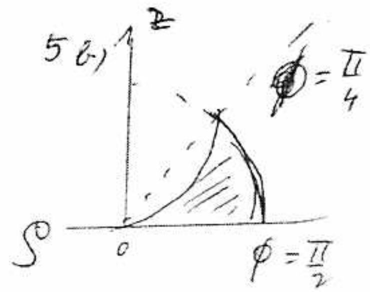
4.



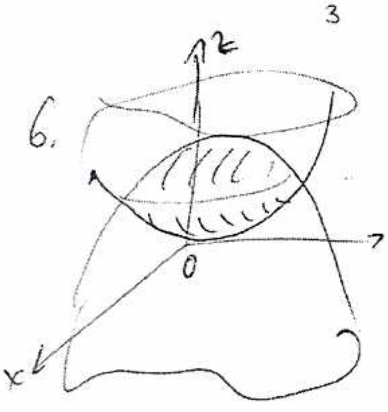
5a)

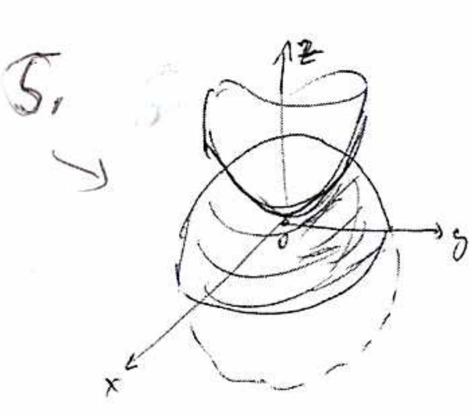


5b)



6.





7. D :