The University of Calgary
Faculty of Science
Department of Mathematics and Statistics
MATH 353 -02 Midterm Test

Time:90 minutes **Winter 2008**

Each questions is for 10 marks.

1. For $f(x,y) = 3x - x^3 - xy^2$ find all local extrema .
   solve $\nabla f = 0$ $\quad f_x = 3 - 3x^2 - y^2 = 0 \quad f_y = -2xy = 0$
   from the second equation two cases:
   $x = 0 \to\to y^2 = 3 \quad \left(0, \pm\sqrt{3}\right)$
   $y = 0 \to\to 3x^2 = 3 \quad (\pm 1, 0)$
   second der.Test:
   $f_{xx} = -6x \quad f_{xy} = -2y \quad f_{yy} = -2x$
   and $D = 4y^2 - 12x^2$ for $\left(0, \pm\sqrt{3}\right)$ $\quad D > 0$ saddle points
   for $(1,0)$ $\quad D < 0, A < 0$ local maximum
   for $(-1,0)$ $\quad D < 0, A > 0$ local minimum

2. For $f(x,y) = 3x - x^3 - xy^2$ find the absolute extrema on the tringle with vertices $(0,0), (2,2)$ and $(2,0)$.
   the set $T = \{0 \le x \le 2, 0 \le y \le x\}$
   using problem 1:inside the triangle no points
   and on the boundary $(1,0)$
   generally 3 parts of the **boundary**: $B_1 = \{0 \le x \le 2, 0 = y\}$
   then f on $B_1 = g(x) = 3x - x^3$ $\quad g'x) = 3(1 - x^2) = 0 \quad (1,0)$
   and ends $(0,0)$ and $(2,0)$
   $B_2 = \{x = 2, 0 \le y \le 2\}$ then f on $B_2 = h(y) = -2(1 + y^2)$ only ends $(2,2)$
   $B_3 = \{0 \le x \le 2, y = x\}$ then f on $B_3 = l(x) = 3x - 2x^3$
   $l'(x) = 3 - 6x^2 = 0 \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
   now values: $\quad f(1,0) = 2$... maximum
   $f(0,0) = 0 \quad f(2,0) = -2 \quad f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$
   $f(2,2) = -10$...minimum

3. $$\iint\limits_D \frac{x}{1-xy}\,dxdy = \int_0^1 \left(\int_0^1 \frac{x}{1-xy} dy \right) dx = (u = 1 - xy, du = -xdy) =$$

$$= \int_0^1 \left[-\ln(1-xy)\right]_{y=0}^{y=1} dx = \int_0^1 -\ln(1-x)dx = (u = 1-x, du = -dx) =$$

$$= \int_1^0 \ln u \; du = -\left[u \ln u - u\right]_0^1 = 1$$

it is convergent since

$$\lim_{u\to 0^+} u\ln u = \text{"}0\cdot -\infty\text{"} \lim_{u\to 0^+} \frac{\ln u}{u^{-1}} = (\text{L'H.R.}) \lim_{u\to 0^+} \frac{u^{-1}}{-u^{-2}} = \lim_{u\to 0^+} -u = 0$$

4. in the coordinates x, y, z

   (a) if in $S = \{\rho = 2, \phi = \frac{3}{4}\pi, \theta \in [0, 2\pi)\} \to z = 2\cos\frac{3}{4}\pi = -\sqrt{2}$
   and $x = 2\cos\theta\sin\frac{3}{4}\pi \qquad y = 2\sin\theta\sin\frac{3}{4}\pi \qquad$ so
   $x = \sqrt{2}\cos\theta \qquad y = \sqrt{2}\sin\theta$
   circle $x^2 + y^2 = 2$, radius $=\sqrt{2}$ on the horiz. plane $z = -\sqrt{2}$

   (b) if in $S = \{z = 2, r \geq 0, \theta = \frac{\pi}{2}\} \qquad x = 0, z = 2, y = r \geq 0$
   $\cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1 \qquad (0, r, 2)$
   a ray = half line parallel to the $y-$ axis in the $yz-$ plane

5. By cyl.coord. $\iiint\limits_B \frac{xz}{\sqrt{x^2+y^2}} \; dxdydz = I$

   where $B = \{(x,y,z)\,; x^2+y^2+z^2 \leq 2, z \geq 0, x \geq 0, x^2+y^2 \geq 1\}$
   inside the sphere ,outside the cylinder

   $I = \iiint\limits_{B^*} \frac{r\cos\theta \;\cdot z}{r} r\; dr d\theta dz$ where $B^* = \{(r,\theta,z)\,; r^2+z^2 \leq 2, z \geq 0, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], r \geq 1\}$

   thus $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \iint\limits_{D^*} rz dr dz$ where $D^* = \{(r,z)\,; r^2+z^2 \leq 2, z \geq 0, r \geq 1\} = \{1 \leq r \leq \sqrt{2}, 0 \leq z \leq \sqrt{2-r^2}\}$

   so $I = 2\left[\sin\theta\right]_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \left(r \int_0^{\sqrt{2-r^2}} z dz \right) dr = 2\int_1^{\sqrt{2}} r\left[\frac{z^2}{2}\right]_0^{\sqrt{2-r^2}} dr = \int_1^{\sqrt{2}} r\left(2-r^2\right) dr = \left[r^2 - \frac{r^4}{4}\right]_1^{\sqrt{2}} =$
   $2 - 1 - 1 + \frac{1}{4} = \dfrac{1}{4}$

   also :$0 \leq z \leq 1, 1 \leq r \leq \sqrt{2-z^2}$

6. from $\int_0^1 \left(\int_z^1 (\int_0^x e^{x^3} dy) dx \right) dz$ given: $0 \leq z \leq 1, z \leq x \leq 1, 0 \leq y \leq x$

   outside x: $0 \leq x \leq 1$ and then $0 \leq y \leq x, 0 \leq z \leq x$ thus

   $$\int_0^1 \left(\int_z^1 (\int_0^x e^{x^3} dy) dx \right) dz = \int_0^1 e^{x^3} \left(\int_0^x (\int_0^x dy) dz \right) dx = \int_0^1 x^2 e^{x^3} dx = \left[\frac{1}{3}e^{x^3}\right]_0^1 = \frac{1}{3}(e-1)\,.$$

7. the integral $I = \iiint\limits_B y \; dxdydz$ where $B = \{(x, y, z)\, ; x^2 + y^2 + z^2 \leq 4, z \geq 0, y \geq 0\}$

in cylindrical $I = \iiint\limits_{B^*} r \sin\theta \cdot r \; drd\theta dz$ where $B^* = \{r^2 + z^2 \leq 4, z \geq 0, \theta \in [0, \pi]\}$

so $$I = \int_0^{\pi} \sin\theta d\theta \int_0^{2} r^2 \left(\int_0^{\sqrt{4-r^2}} dz \right) dr = [-\cos\theta]_0^{\pi} \cdot \int_0^{2} r^2\sqrt{4 - r^2} dr(\text{ Table})...$$

in spherical coordinates $I = \iiint\limits_{B^{**}} \rho \sin\theta \sin\phi \cdot \rho^2 \sin\phi \; d\rho d\theta d\phi$

where $B^{**} = \{0 \leq \rho \leq 2, \phi \in \left[0, \frac{\pi}{2}\right], \theta \in [0, \pi]\}$

thus

$$I = \int_0^{\pi} \sin\theta d\theta \int_0^{\frac{\pi}{2}} \sin^2\phi \; d\phi \int_0^{2} \rho^3 d\rho = 2\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\phi}{2} d\phi \left[\frac{\rho^4}{4}\right]_0^2 = \left[\tfrac{\pi}{2} - 0\right] \cdot 4 = 2\pi.$$