The University of Calgary
Department of Mathematics and Statistics
MATH 353-02
Quiz #5T(10am)

Winter 2008

1. Evaluate $\iint\limits_S x^2 z \, dS$ and S is the top($z > 0$) part of the cylinder $9 = y^2 + z^2$ that lies inside the cylinder $x^2 + y^2 = 9$. [3]

2. Find $\iint\limits_S \mathbf{F} \bullet \mathbf{dS}$ where $\mathbf{F}(x, y, z) = (xz, yz, x^2 + y^2 + z^2)$ and S is the part of the plane $2 = x + y$ below the plane $3x + 2y + z = 6$ in the first octant oriented in the direction of positive x-axis. [4]

3. For $\mathbf{F}(x, y, z) = (\sqrt{x^2 + zy^2}, \ln(2z + y^2 + 1), \frac{y}{z})$ find $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F}$ for $z > 0, x, y$ any. [3]

SOLUTION

For 1)

S is given by $z = \sqrt{9 - y^2}$ for $(x, y) \in D = \{x^2 + y^2 \leq 9\}$

$$\mathbf{n} = \left(0, \frac{-y}{\sqrt{9 - y^2}}, -1\right) \qquad \|\mathbf{n}\| = \sqrt{\frac{y^2}{9 - y^2} + 1} = \frac{3}{\sqrt{9 - x^2}}$$

f on S $\quad \rightarrow$

$$\iint\limits_S x^2 z \, dS = \iint\limits_D x^2 \sqrt{9 - y^2} \frac{3}{\sqrt{9 - y^2}} \, dxdy = 3 \iint\limits_D x^2 \, dxdy$$

(by polar.coord.)

$$= 3 \int_0^{2\pi} \int_0^3 r^2 \cos^2 \theta \, r dr d\theta = 3 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta \int_0^3 r^3 dr = 3\pi \frac{3^4}{4} = \frac{3^5}{4} \pi.$$

For 2)

below $\rightarrow 0 \leq z \leq 6 - 3x - 2y$

S is a part of vertical surface $\rightarrow$ parametrization

we can choose $\quad x = u \quad y = 2 - u \quad z = v$

$\mathbf{r}(u, v) = (u, 2 - u, v)$ for $u \in [0, 2]$and $0 \leq v \leq 6 - 3u - 4 + 2u = 2 - u$

$$\frac{\partial \mathbf{r}(u, v)}{\partial u} = (1, -1, 0) \qquad \frac{\partial \mathbf{r}(u, v)}{\partial v} = (0, 0, 1)$$

cross product $\quad \mathbf{n} = +(1, 1, 0)$ - positive x-axis

$\mathbf{F}$ on S $\quad \mathbf{F}(x,y,z) = (uv, (2-u)\,v, ...)$and $\mathbf{F} \bullet \mathbf{n} = 2v$

$$\iint\limits_S \mathbf{F} \bullet \mathbf{dS} = \iint\limits_D \mathbf{F} \bullet \mathbf{n}\; dudv \text{ where } D\left\{(u,v)\,;u \in [0,2]\,, 0 \le v \le 2-u\right\}$$

$$= \int_0^2 \left(\int_0^{2-u} 2vdv \right) du = \int_0^2 (2-u)^2 du = \left[\frac{(u-2)^3}{3}\right]_0^2 = \frac{8}{3}.$$

For 3)

$$\operatorname{div} \mathbf{F} = \partial_x \left(\sqrt{x^2+zy^2}\right) + \partial_y \ln(2z+y^2+1) + \partial_z \frac{y}{z} = \frac{x}{\sqrt{x^2+zy^2}} + \frac{2y}{2z+y^2+1} - \frac{y}{z^2}$$

$$\mathbf{curl}(\mathbf{F}) = \begin{vmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \sqrt{x^2+zy^2} & \ln(2z+y^2+1) & \dfrac{y}{z} \end{vmatrix} = \left(\frac{1}{z} - \frac{2}{2z+y^2+1}, \frac{y^2}{2\sqrt{x^2+zy^2}}, -\frac{yz}{\sqrt{x^2+zy^2}} \right)$$