Handout 1 - Solutions

4. = 13.1-3 $\quad f(x,y) = x^3 + y^3 - 3xy \quad D = \mathbb{R}^2$

$f_1 = 3x^2 - 3y = 0$

$f_2 = 3y^2 - 3x$

$$Hf(x,y) = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} = 3\begin{bmatrix} 2x & -1 \\ -1 & 2y \end{bmatrix}$$

$$\vec{\nabla} f = 0 \implies \begin{cases} x^2 = y \\ y^2 = x \end{cases} \implies x = y^2 = x^4 \implies x(x^3-1) = 0 \implies x = 0, 1$$

Gives $P = (0,0)$, $Q = (1,1)$ crit pts

$Hf(P) = 3\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ indefinite $\implies$ saddle pt at $P = (0,0)$

$Hf(Q) = 3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ + def $\implies$ local min at $Q = (1,1)$

5. = 13.1-15 $\quad f(x,y) = (1+\frac{1}{x})(1+\frac{1}{y})(\frac{1}{x}+\frac{1}{y}) \quad D = \{(x,y) : x \neq 0, y \neq 0\}$.

After a little simplification

$f_1 = -(1+\frac{1}{y})(\frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^2 y})$

$f_2 = -(1+\frac{1}{x})(\frac{1}{y^2} + \frac{2}{y^3} + \frac{1}{xy^2})$

$f_{11} = -(1+\frac{1}{y})(\frac{-2}{x^3} - \frac{6}{x^4} - \frac{2}{x^3 y})$

$f_{12} = \frac{1}{y^2}(\frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^2 y}) + (1+\frac{1}{y})\frac{1}{x^2y^2}$

$f_{22} = -(1+\frac{1}{x})(-\frac{2}{y^3} - \frac{6}{y^4} - \frac{2}{x^3 y})$

$f_1 = 0 \implies y = -1$ or $xy + 2y + x = 0$ so $P = (-1,-1)$ is a critical pt

$f_2 = 0 \implies x = -1$ or $xy + 2x + y = 0$

$y = -1$ and $xy + 2x + y = 0 \implies x = 1 \implies Q = (1,-1)$ is a critical pt

Similarly $R = (1,-1)$ is a crit pt

Finally $\begin{cases} xy + 2y + x = 0 \\ xy + 2x + y = 0 \end{cases} \implies x = y$ (by subtraction)

$\implies x^2 + 3x = 0 \implies x = -3$ ($x=0$ not in D)

So $S = (-3,-3)$ is the final crit pt

$Hf(P) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$, $Hf(Q) = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}$, $Hf(R) = \begin{bmatrix} -4 & 2 \\ 2 & 0 \end{bmatrix}$ all indefinite all saddle pts

After some arithmetic, $Hf(S) = \frac{2}{243}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ + def $\implies$ local min at $S = (-3,-3)$.