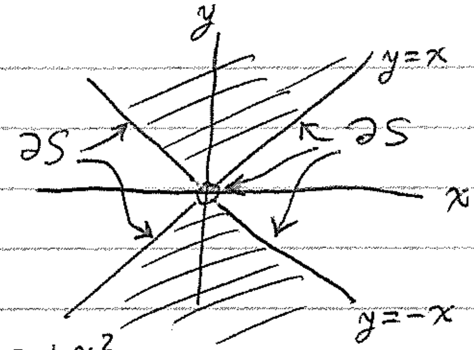
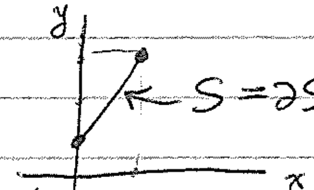
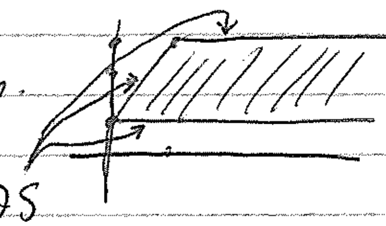
Handout 1 – Solutions



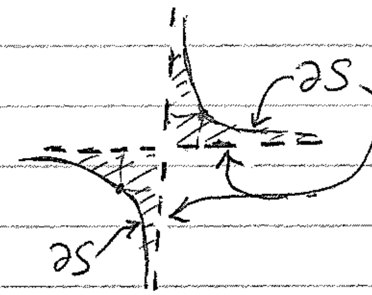
1. (a) S = shaded area, including the two lines $y = \pm x$, but excluding $(0,0)$.
S is not bounded – clear
S is not open since part of ∂S is contained in S. $\partial S = \{(x,y) : y = \pm x\}$.
S is not closed since one point of ∂S, $(0,0)$, is not in S.



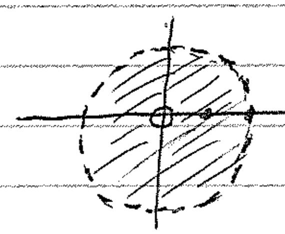
(b) $S = \{(x,y) : y - 2x = 1,\ 1 \le y \le 3\}$
is a line segment.
$S = \partial S$, S is closed and bounded (compact), not open.



(c) S = shaded region, closed, not bounded, not open.
$\partial S = \{(x,y) : x \ge 1, y = 3\} \cup \{(x,y) : y - 2x = 1, 1 \le y \le 3\}$
$\cup \{(x,y) : x \ge 0,\ y = 1\}$



(d) $S = \{(x,y) : \ln(xy) \le 0\} = \{(x,y) : 0 < xy \le 1\}$
= shaded region, where x, y axes excluded.
$\partial S = \{(x,y) : xy = 1\} \cup \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$.
S is not bounded, not open, not closed.



(e) S = shaded area, where $(0,0)$ and the circle $x^2 + y^2 = 4$ are excluded.
$\partial S = \{(0,0)\} \cup \{(x,y) : x^2 + y^2 = 4\}$
S is bounded, open, not closed

(f) Similar to (e), but the circle $x^2 + y^2 = 4$ now is included in S. ∂S same, now S is bounded, but not open, not closed.

2. $f(x,y) = 2xy^2 - x^2y + 4xy$ Defined and differentiable on all of $\mathbb{R}^2$.

$$\left.\begin{array}{l} f_x = 2y^2 - 2xy + 4y \\ f_y = 4xy - x^2 + 4x \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} f_{xx} = -2y \\ f_{xy} = 4y - 2x + 4 = f_{yx} \\ f_{yy} = 4x \end{array}\right.$$

$$\vec{\nabla} f = 0 \Longrightarrow \begin{array}{l} 0 = f_x = 2y(y - x + 2) \\ 0 = f_y = x(4y - x + 4) \end{array} \Longrightarrow \left\{\begin{array}{l} y = 0 \text{ or } y - x + 2 = 0 \\ x = 0 \text{ or } 4y - x + 4 = 0 \end{array}\right.$$

Solving these - be careful to account for all possibilities - gives four critical points $P = (0,0)$, $Q = (0,-2)$, $R = (4,0)$, $S = (\frac{4}{3}, -\frac{2}{3})$.

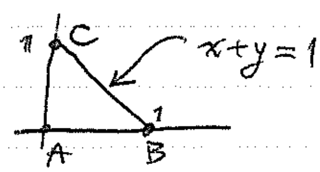
$$H(x,y) = \begin{bmatrix} -2y & 4y-2x+4 \\ 4y-2x+4 & 4x \end{bmatrix}$$

$H(P) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ indefinite $\Longrightarrow P =$ saddle pt, similarly for Q, R.

$H(S) = \begin{bmatrix} \frac{4}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{16}{3} \end{bmatrix}$ easier to use $(\frac{3}{4})S = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$ $P_1 = 1 > 0$, $P_2 = 3 > 0$

$\Longrightarrow$ + def $\Longrightarrow$ local min at S.

Since $f(1,y) = 2y^2 + 3y$ approches $+\infty$ as $y \to +\infty$, and $f(-1,y) = -2y^2 - 5y$ approaches $-\infty$ as $y \to \infty$, f has no abs max or abs min.



Now consider f on the three edges $AB \cup BC \cup CA$. Since this is a compact set there must be an abs max & abs min.

On AB and on AC, $f(x,y) = 0$.

On BC, $y = 1-x$, so $f(x,y) = f(x, 1-x) = g(x)$, $0 \le x \le 1$.

$= 2x(1-x)^2 - x^2(1-x) + 4x(1-x) = x(1-x)(2-2x-x+4)$

$g(x) = x(1-x)(6-3x) = 3x(1-x)(2-x) = 3(x^3 - 3x^2 + 2x)$

$0 = g'(x) = 3(3x^2 - 6x + 2) \Longrightarrow x = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$

Since $1 + \sqrt{3}/3 > 1$ it is discarded.

At $x = 1 - \frac{\sqrt{3}}{3}$ (which satisfies $0 \le x \le 1$), we find $g(x) = \frac{2}{3}\sqrt{3}$ giving the abs max at $(1 - \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, abs min $= 0$ along $AB \cup AC$

3. $f(x,y) = 3y^3 - x^2y + x^2$ Domain all of $\mathbb{R}^2$

$\vec{\nabla} f = 0 \implies -2xy + 2x = 0,\ 9y^2 - x^2 = 0$

gives three critical pts $P = (0,0)$, $Q = (-3,1)$, $R = (3,1)$.

One finds $H(Q) = \begin{bmatrix} 0 & 6 \\ 6 & 18 \end{bmatrix}$ indef, $H(R) = \begin{bmatrix} 0 & -6 \\ -6 & 18 \end{bmatrix}$ indef,

so Q, R are saddle pts.

$H(P) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ has determinant $= 0$, so H-test fails.

Since $f(0,y) = 3y^3$, we see that no local max or min at P.

4-5 somewhat similar to 2, 3 and answers given in text.

Note in #5 the domain is $D = \{(x,y) : x \neq 0 \text{ and } y \neq 0\}$.

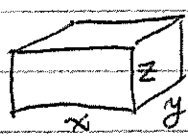
6. $f(x,y,z) = xy + x^2z - x^2 - y - z^2$, $D = \mathbb{R}^3$

One easily finds $\vec{\nabla} f = \langle y + 2xz - 2x,\ x - 1,\ x^2 - 2z \rangle$,

$$H(x,y,z) = \begin{bmatrix} 2z-2 & 1 & 2x \\ 1 & 0 & 0 \\ 2x & 0 & -2 \end{bmatrix}, \quad \text{and } \vec{\nabla} f = 0 \implies x=1, y=1, z=\tfrac{1}{2}.$$

So only one critical point $P = (1, 1, \frac{1}{2})$.

$$H(P) = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \quad \left.\begin{matrix} P_1 = -1 < 0 \\ P_2 = -1 < 0 \\ P_3 = 2 > 0 \end{matrix}\right\} \implies \text{indefinite} \implies P = \text{saddle pt}$$

7. 

Maximize $V = xyz$, given $z + 2x + 2y \leq L$.

If $z + 2x + 2y < L$, then one can increase all of z, x, y so that $z + 2x + 2y < L$ is still true, and V has increased. So for V_{max}, $z + 2x + 2y = L$. $z = L - 2x - 2y$

$V = xyz = xy(L - 2x - 2y) = Lxy - 2x^2y - 2xy^2$, where now we have (x,y) in the domain given by $0 < x, y < L/2$ (we exclude the boundary of this domain since $V = 0$ on the bndry).

$\begin{cases} Ly - 4xy - 2y^2 = 0 \\ Lx - 2x^2 - 4xy = 0 \end{cases} \implies (x,y) = P = (\frac{L}{6}, \frac{L}{6})$, only critical pt in the domain. Then $z = \frac{L}{3}$, $V_{max} = L^3/108$.

Remark: It's also worth it to do this question via Lagrange multipliers.