Solution for handout #2.

Q#1 Solution: The function is Continuous and the disk $x^2+y^2 \le 65$ is bounded. and closed. So the absoute extrema of $f(x,y)$ exist.

Solve $\nabla f = \langle 0,0 \rangle$. we get the first critical point $(0,0)$.

For critical points on the boundary, we use Lagrange multiplier method, where $g(x,y) = x^2+y^2-65$.

Define $L(x,y,\lambda) = \frac{1}{8}x^3 + y^3 + \lambda(x^2+y^2-65)$. Solve $\nabla L = \langle 0,0,0 \rangle$.

we obtain $\begin{cases} \frac{3}{8}x^2 + 2\lambda x = 0 \longrightarrow x=0 \text{ or } \frac{3}{8}x = -2\lambda \\ 3y^2 + 2\lambda y = 0 \longrightarrow y=0 \text{ or } 3y = -2\lambda \\ x^2+y^2-65 = 0 \end{cases}$

If $x=0$, $y=\pm\sqrt{65}$. $\longrightarrow$ two critical points $(0,\sqrt{65})$; $(0,-\sqrt{65})$.

similarly, if $y=0$, $x=\pm\sqrt{65} \longrightarrow$ two critical points $(\sqrt{65},0)$; $(-\sqrt{65},0)$.

If $x\neq 0$, $y \neq 0$, $\frac{3}{8}x = 3y \Rightarrow x = 8y$. then from $x^2+y^2=65$.

we get $64y^2+y^2 = 65 \Rightarrow y^2=1 \Rightarrow y=\pm 1$.

if $y=1$, $x=8$, if $y=-1$, $x=-8$. $\Rightarrow$ two critical points $(-8,-1)$ & $(8,1)$

Now we evaluate $f(x,y)$ at these points.

$f(0,0)=0$, $f(0,\sqrt{65}) = \frac{1}{8}\cdot 0 + (\sqrt{65})^3 = 65\sqrt{65}$, $f(0,-\sqrt{65}) = -65\sqrt{65}$.

$f(\sqrt{65},0) = \frac{1}{8}65\sqrt{65} = \frac{65}{8}\sqrt{65}$, $f(-\sqrt{65},0) = -\frac{65}{8}\sqrt{65}$;

$f(-8,-1) = \frac{1}{8}(-8)^3-1 = -65$; $f(8,1) = \frac{1}{8}\cdot 8^3+1 = 65$.

So the absolute Max. is $65\sqrt{65}$. which is obtained at $(0,\sqrt{65})$.

while the absolute Min is $-65\sqrt{65}$, which is obtained at $(0,-\sqrt{65})$.

Q#2. (a): D_a is closed and bounded. So it is Compact.

(b): D_b is closed and bounded. So it is compact.

(c): D_c is not bounded. thus it is not compact.

for example, let $z=2$, then $x=\pm 2y$, both x and y can be $\pm\infty$.

(d): D_d is not bounded. thus it is NOT Compact.

(e): D_e is bounded and closed, so it is compact.

Q#3. Solution: Assume the minimum distance is obtained at the point (x,y,z). then we are looking for the min. of $d(x,y,z)$, which is defined as $d(x,y,z) = \sqrt{(x-1)^2+(y+1)^2+(z-2)^2}$.

To simplify the calculation, we consider the square of $d(x,y,z)$.

Thus we want to solve the problem

minimize $f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-2)^2$ subject to $g(x,y,z) = x-2y-z-3=0$.

Define $L(x,y,z,\lambda) = (x-1)^2 + (y+1)^2 + (z-2)^2 + \lambda(x-2y-z-3)$.

For critical points of $L(x,y,z,\lambda)$ we need

$L_x = 2(x-1) + \lambda = 0 \longrightarrow x = -\frac{\lambda}{2} + 1$ ①

$L_y = 2(y+1) - 2\lambda = 0 \longrightarrow y = \lambda - 1$ ②

$L_z = 2(z-2) - \lambda = 0 \longrightarrow z = \frac{\lambda}{2} + 2$ ③

$L_\lambda = x-2y-z-3 = 0$. ④

Substitute ①, ② and ③ into ④. $-\frac{\lambda}{2} + 1 - 2\lambda + 2 - \frac{\lambda}{2} - 2 - 3 = 0$

$\Rightarrow -3\lambda = 2 \Rightarrow \lambda = -\frac{2}{3}$. then $x = \frac{4}{3}$, $y = -\frac{5}{3}$, $z = \frac{5}{3}$.

thus the minimum distance is $\sqrt{(\frac{4}{3}-1)^2 + (-\frac{5}{3}+1)^2 + (\frac{5}{3}-2)^2} = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1$.

To justify that we have found minimum.

The set is NOT bounded, but if we take a bounded part we have to have absolute maximum and minimum. However, the maximum will be on the boundary since the distance is increasing when we move far away, therefore the critical point must be minimum.

Q#4. Assume that the diagonally opposite vertex is at (x,y,z) the volume is maximized, so we need to solve this problem:

Minimize $f(x,y,z) = xyz$ subject to $xy + 2yz + 3xz - 18 = 0$. $x,y,z \geq 0$

let $L(x,y,z,\lambda) = xyz + \lambda(xy + 2yz + 3xz - 18)$,

For critical point(s) of $L(x,y,z)$, we need

$$\begin{cases} L_x = yz + \lambda y + 3\lambda z = 0 & ① \\ L_y = xz + \lambda x + 2\lambda z = 0 & ② \\ L_z = xy + 2\lambda y + 3\lambda x = 0 & ③ \\ L_\lambda = xy + 2yz + 3xz - 18 = 0 & ④ \end{cases}$$

Obviously, $x \neq 0$, $y \neq 0$, $z \neq 0$, otherwise the volume $f(x,y,z) = 0$

From ①, $yz = -\lambda(y+3z)$; from ②, $xz = -\lambda(x+2z)$; from ③, $xy = -\lambda(2y+3x)$.

So $-\frac{1}{\lambda} = \frac{y+3z}{yz} = \frac{x+2z}{xz} = \frac{2y+3x}{xy}$.

From the first equality (cancel z). $\Rightarrow xy + 3xz = xy + 2zy \Rightarrow x = \frac{2}{3}y$

From the second equality (cancel x) $\Rightarrow xy+2yz = 2yz+3xz \Rightarrow z = \frac{1}{3}y$.

Substitute the results back to the surface:

$\frac{2}{3}y\cdot y + 2y\cdot\frac{1}{3}y + 3\cdot\frac{2}{3}y\cdot\frac{1}{3}y = 18 \Rightarrow (\frac{2}{3}+\frac{2}{3}+\frac{2}{3})y^2 = 18 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$.

drop $y=-3$. So $y=3$. then $x = \frac{2}{3}y = 2$. $z = \frac{1}{3}y = 1$.

So the maximum volume of the box is $V_{max} = 2\cdot 3\cdot 1 = 6$

Q#5. Use the method of Lagrange multiplier for multiple constraints.

Let $L(x,y,z,\lambda,u) = x + y^2 z + \lambda(y^2+z^2-2) + u(z-x)$

For the critical point(s) of $L(x,y,z,\lambda,u)$, we need

$$\begin{cases} L_x = 1-u = 0 \rightarrow u = 1. & ① \\ L_y = 2yz + 2\lambda y = 0 & ② \\ L_z = y^2 + 2\lambda z + u = 0 & ③ \\ L_\lambda = y^2+z^2-2 = 0 & ④ \\ L_u = z - x = 0 \rightarrow z = x. & ⑤ \end{cases}$$

From ②, $y(z+\lambda) = 0 \rightarrow y=0$ or $z = -\lambda$.

If $y=0$, then $z^2 = 2 \Rightarrow z = \pm\sqrt{2}$, thus $x = \pm\sqrt{2}$.

So we have two critical points $(\sqrt{2}, 0, \sqrt{2})$ and $(-\sqrt{2}, 0, -\sqrt{2})$

If $y \neq 0$, then $\lambda = -z$, then from ③, $y^2 + 2\cdot(-z)z + 1 = 0$

$\Rightarrow y^2 - 2z^2 = -1$, Combine with ④, we have $y^2 = 1$ and $z^2 = 1$.

So we have 4 critical points: $(1,1,1)$; $(-1,1,-1)$, $(1,-1,1)$; $(-1,-1,-$

evaluate $f(x,y,z)$ at these points.

$f(\sqrt{2},0,\sqrt{2}) = \sqrt{2}$; $f(-\sqrt{2},0,-\sqrt{2}) = -\sqrt{2}$

$f(1,1,1) = 2$; $f(-1,1,-1) = -2$; $f(1,-1,1) = 2$, $f(-1,-1,-1) = -2$.

So the maximum value is 2, which is obtained at $(1,1,1)$ and $(1,-1,1$

while the minimum value is -2, which is obtained at $(-1,1,-1)$ and $(-1,-1,-1$

Q#6. Use the method of Lagrange multiplier for inequality constraint.

Let $L(x,y,z,\lambda,u) = xy + z^2 + \lambda(x^2+y^2+z^2-1+u^2)$

For the critical points of $L(x,y,z,\lambda,u)$, we need

$L_x = y + 2\lambda x = 0$ ①

$L_y = x + 2\lambda y = 0$ ②

$L_z = 2z + 2\lambda z = 0$ ③

$L_\lambda = x^2+y^2+z^2-1+u^2 = 0$ ④

$L_u = 2\lambda u = 0$ ⑤

From ⑤, If $u \neq 0$, $\lambda = 0$, then from ①, $y=0$, then from ② $x=0$, and from ③, $z=0$ so $(0,0,0)$ is a critical point.

If $u = 0$, from ④, $x^2+y^2+z^2=1$.

from ①: $y = -2\lambda x$. substitute it into ②: $x + 2\lambda \cdot (-2\lambda x) = 0$

$\Rightarrow x(1-4\lambda^2) = 0 \Rightarrow x = 0$ or $1 - 4\lambda^2 = 0$.

If $x=0$, then $y=0$, thus $z^2 = 1 \Rightarrow z = \pm 1$.

$\Rightarrow$ two critical points $(0,0,1)$ and $(0,0,-1)$.

If $x \neq 0$, then $1-4\lambda^2 = 0 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$.

then from ③ $z=0$.

If $\lambda = \frac{1}{2}$, from ② $x + 2\cdot\frac{1}{2}y = 0 \Rightarrow x = -y$. $\Rightarrow x^2+y^2 = 2x^2 = 1 \Rightarrow x^2 = 1/2$

so $x = \pm\frac{\sqrt{2}}{2}$, $\Rightarrow$ two critical points $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$.

If $\lambda = -\frac{1}{2}$. from ②: $x + 2\cdot(-\frac{1}{2})y = 0 \Rightarrow x = y. \Rightarrow x^2+y^2 = 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$.

So $x = \pm\frac{\sqrt{2}}{2} \Rightarrow$ two critical points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$

Evaluate $f(x,y,z)$ at these critical points:

$f(0,0,0) = 0$, $f(0,0,1) = 1$; $f(0,0,-1) = 1$;

$f(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0) = -\frac{1}{2}$; $f(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) = -\frac{1}{2}$;

$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) = \frac{1}{2}$; $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0) = \frac{1}{2}$.

So the Maximum is 1, which is obtained at $(0,0,1)$ and $(0,0,-1)$

While the minimum is $-\frac{1}{2}$, which is obtained at $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$

Q#7(a): $\int_1^3 \int_{-x}^{x^2} x e^{2y}\,dy\,dx = \int_1^3 \frac{1}{2} x e^{2y}\Big|_{y=-x}^{y=x^2} dx = \frac{1}{2}\int_1^3 x e^{2x^2} - x e^{-2x}\,dx$

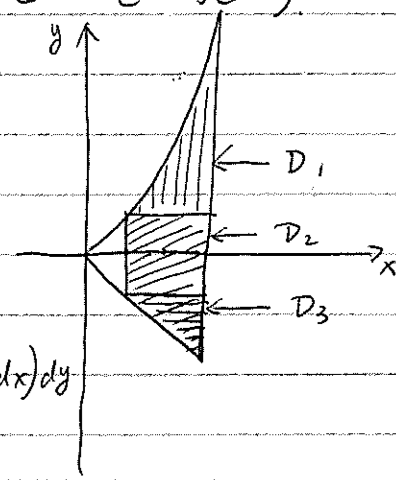
$= \frac{1}{2}\left[\frac{1}{4}e^{2x^2} + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x}\right]\Big|_{x=1}^{x=3} = \frac{1}{8}(e^{18} + 7e^{-6} - e^2 - 3e^{-2})$

(b) we split the domain D into $D = D_1 \cup D_2 \cup D_3$.

where $D_1 = \{(x,y): 1 \le y \le 9,\ \sqrt{y} \le x \le 3\}$.

$D_2 = \{(x,y): -1 \le y \le 1,\ 1 \le x \le 3\}$.

$D_3 = \{(x,y): -3 \le y \le -1,\ -y \le x \le 3\}$.

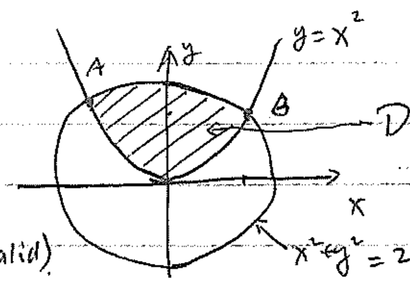


then $I = \iint_{D_1} xe^{2y}\,dA + \iint_{D_2} xe^{2y}\,dA + \iint_{D_3} xe^{2y}\,dA$.

$= \int_1^9 \int_{\sqrt{y}}^3 xe^{2y}\,dx\,dy + \int_{-1}^1 \left(\int_1^3 xe^{2y}\,dx\right)dy + \int_{-3}^{-1}\left(\int_{-y}^3 xe^{2y}\,dx\right)dy$

$= \frac{1}{8}(e^{18} - 7e^{-6} - e^2 + 3e^{-2})$.

Q#8. We first find the two intersection points of the curves $y=x^2$ and $x^2+y^2=2$.



Solve $\begin{cases} y=x^2 \\ x^2+y^2=2, \end{cases}$ we have $y=1$ or $y=-2$ (Not valid)

If $y=1$, $x=\pm 1$. So the two points are $A:(-1,1)$ and $B(1,1)$.

So the domain D is y-simple and it is bounded by $x=-1$, $x=1$, $y=x^2$ and $y=\sqrt{2-x^2}$.

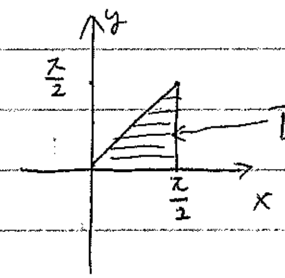
then $\iint_D \sqrt{2-x^2}\,dA = \int_{-1}^{1} dx \int_{x^2}^{\sqrt{2-x^2}} \sqrt{2-x^2}\,dy = \int_{-1}^{1} \sqrt{2-x^2}\left(\sqrt{2-x^2}-x^2\right)dx.$

let $x=\sqrt{2}\sin t$, $t\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$. then $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{2}\cdot\cos t\left(\sqrt{2}\cos t-2\sin^2 t\right)\sqrt{2}\cos t\,dt$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\sqrt{2}\cos^3 t - 4\sin^2 t\cos^2 t\,dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\sqrt{2}\left(1-\sin^2 t\right)d\sin t - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 2t\,dt$

$= 2\sqrt{2}\left(\sin t-\frac{1}{3}\sin^3 t\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\left(1-\cos 4t\right)dt = \frac{10}{3}-\frac{\pi}{4}.$

Q#9. Obviously, it is difficult to integrate $\frac{\sin x}{x}$. So we change the order of integration. The domain D is enclosed by $y=0$, $y=\frac{\pi}{2}$, $x=\frac{\pi}{2}$ and $x=y$.

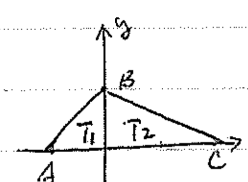


So $\int_0^{\frac{\pi}{2}}\int_y^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx\,dy = \iint_D \frac{\sin x}{x}\,dA$

$= \int_0^{\frac{\pi}{2}}\int_0^{x} \frac{\sin x}{x}\,dy\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\cdot(x-0)\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx$

$= -\cos x\Big|_0^{\frac{\pi}{2}} = 1$

Q#10. By the definition the volume $V=\iint_T e^{(y-1)^2}dA$, where T is the triangle with vertices A, B and C, which lies in the x-y plane.



We split T into $T=T_1\cup T_2$, where $T_1=\{(x,y): 0\le y\le 1,\ y-1\le x\le 0\}$

$T_2=\{(x,y): 0\le y\le 1,\ 0\le x\le 2(1-y)\}$. So $V=\iint_{T_1} e^{(y-1)^2}dA + \iint_{T_2} e^{(y-1)^2}dA$

$= \int_0^1\int_{y-1}^{0} e^{(y-1)^2}dx\,dy + \int_0^1\int_0^{2(1-y)} e^{(y-1)^2}dx\,dy = -\int_0^1 (y-1)e^{(y-1)^2}dy - 2\int_0^1 (y-1)e^{(y-1)^2}dy$

$= -3\int_0^1 (y-1)e^{(y-1)^2}dy \overset{\text{let } u=y-1}{=\!=\!=} -3\int_{-1}^{0} ue^{u^2}du = -\frac{3}{2}e^{u^2}\Big|_{-1}^{0} = -\frac{3}{2}(1-e) = \frac{3}{2}(e-1).$