**The University of Calgary**
**Department of Mathematics and Statistics**
**MATH 353 Handout #3 Solutions**

1. The intersection of the two surfaces happens when $z = \cos\sqrt{x^2+y^2} = 0$, and this is exactly when $\sqrt{x^2+y^2} = \pi/2$, i.e. $x^2+y^2 = (\pi/2)^2$. So $D = \{(x,y)|x^2+y^2 \le \left(\frac{\pi}{2}\right)^2\}$ and
$$V = \iint_D \cos\sqrt{x^2+y^2}dxdy = \iint_D r\cos r\ drd\theta$$
(polar coord.) where in polar coordinates $D = \{(r,\theta)|0 < r \le \frac{\pi}{2}; 0 \le \theta < 2\pi\}$. Using integration by parts $V = 2\pi\int_0^{\frac{\pi}{2}} r\cos r\ dr = 2\pi\left[r\sin r - \iint \sin r\ de\right]_0^{\frac{\pi}{2}} = \pi^2 + 2\pi\left[\cos r\right]_0^{\frac{\pi}{2}} = \pi^2 - 2\pi$.

2. Using polar coord.
$$\iint_D e^{3(x^2+y^2)}dxdy = \iint_D e^{3r^2} r\ drd\theta = \pi\left[\frac{e^{3r^2}}{6}\right]_1^2 = \frac{\pi}{6}\left(e^{12}-e^3\right)$$
where $D = \{0 \le \theta \le \pi, 1 \le r \le 2\}$

3. The triangle is $T = \{0 \le x \le 2, 2x \le y \le 4\}$ so
$$I = \iint_T \frac{1}{(y-2x)^k}\ dA = \int_0^2 (\int_{2x}^4 (y-2x)^{-k}\ dy)dx = \int_0^2 \left[\frac{(y-2x)^{1-k}}{1-k}\right]_{y\to 2x}^{y=4} dx$$
for $k \ne 1$.
Then $\lim_{y\to 2x}(y-2x)^{1-k} = 0$ for $1-k > 0$ and diverges for $1-k < 0$.
For $k = 1$, the anti-derivative is a logarithm and $[\ln(y-2x)]_{y\to 2x}^{y=4} = +\infty$. So the integral is convergent for only $k < 1$ and
$$I = \int_0^2 \left[\frac{(4-2x)^{1-k}}{1-k}\right] dx = \left[\frac{(4-2x)^{2-k}}{(-2)(1-k)(2-k)}\right]_0^2 = \frac{2}{(1-k)(2-k)}$$
since $2-k > 0$.

4. Use polar coordinates, let $x = r\cos\theta$, $y = r\sin\theta$ then $I = \iint_D \frac{1}{\sqrt{x^2+y^2}}dxdy = \iint_D drd\theta$. The boundary of $D$ in polars can be obtained by replacing $x$ by $r\cos\theta$ and $y$ by $r\sin\theta$:
$x^2+y^2 \le 2 \Longrightarrow r^2 \le 2 \Longrightarrow r \le \sqrt{2}$. $x \ge 1 \Longrightarrow r\cos\theta \ge 1 \Longrightarrow r \ge \frac{1}{\cos\theta}$, $y \ge 0 \Longrightarrow \sin\theta \ge 0 \Longrightarrow \theta \in [0,\pi]$, on the other side, $\frac{1}{\cos\theta} \le r \le 2 \Longrightarrow \frac{1}{\cos\theta} \le 2 \Longrightarrow \cos\theta \ge \frac{1}{\sqrt{2}} \Longrightarrow \theta \in [0, \frac{\pi}{4}]$, Then
$$\begin{aligned} I &= \int_0^{\pi/4}\int_{\frac{1}{\cos\theta}}^{\sqrt{2}} drd\theta = \int_0^{\frac{\pi}{4}}\left(\sqrt{2} - \frac{1}{\cos\theta}\right)d\theta \\ &= \frac{\sqrt{2}\pi}{4} - \left(\ln|sec\theta + tan\theta|\right)_0^{\frac{\pi}{4}} = \frac{\sqrt{2}\pi}{4} - \ln(\sqrt{2}+1). \end{aligned}$$

5. $D = \{x \in (1, +\infty), 0 \le y \le \frac{1}{x^2}\}$ is unbounded so

$$\begin{aligned} I &= \iint_D e^{-x^2 y} dA = \int_1^\infty \left( \int_0^{\frac{1}{x^2}} e^{-x^2 y} dy \right) dx \\ &= \int_1^\infty \left[ \frac{e^{-x^2 y}}{-x^2} \right]_{y=0}^{y=\frac{1}{x^2}} dx = \int_1^\infty \left[ \frac{e^{-1} - 1}{-x^2} \right] dx \\ &= \left( \frac{1}{e} - 1 \right) \left[ \frac{1}{x} \right]_{x=1}^{x \to \infty} = 1 - \frac{1}{e}. \end{aligned}$$

6. The function is unbounded

$$\begin{aligned} I &= \iint_D \frac{1 + \ln x}{y} dA = \int_0^1 \frac{1}{y} ( \int_0^{e^y} (1 + \ln x)\, dx) dy \\ &= \int_0^1 \frac{1}{y} [x \ln x]_0^{e^y} dy = \int_0^1 \frac{1}{y} (e^y y)\, dy \qquad \left( \lim_{\mathbf{x} \to \mathbf{0}+} \mathbf{x} \ln \mathbf{x} = \mathbf{0} \right) \\ &= e - 1 \end{aligned}$$

7.

$$\begin{aligned} \iiint_R yz^2 e^{-xyz} dV &= \int_0^1 dz \int_0^1 dy \int_0^1 yz^2 e^{-xyz} dx \\ &= \int_0^1 dz \int_0^1 (-ze^{-xyz}|_{x=0}^{x=1}) dy = \int_0^1 dz \int_0^1 (-ze^{-yz} + z) dy \\ &= \int_0^1 (zy + e^{-yz})|_{y=0}^{y=1} dz = \int_0^1 \left( z + e^{-z} - 1 \right) dz \\ &= \left( \frac{z^2}{2} - e^{-z} - z \right) |_{z=0}^{z=1} = \frac{1}{2} - \frac{1}{e} \end{aligned}$$

8.

$$\begin{aligned} \iiint_T x dV &= \int_0^1 dz \int_{1-z}^1 dy \int_{2-z-y}^1 x dx \\ &= \int_0^1 dz \int_{1-z}^1 \left( \frac{x^2}{2} \right) |_{x=2-z-y}^{x=1} dy = \int_0^1 dz \int_{1-z}^1 (\frac{1}{2} - \frac{1}{2}(2 - z - y)^2) dy \\ &= \int_0^1 (\frac{y}{2} + \frac{(2 - z - y)^3}{6})|_{y=1-z}^{y=1} dz = \int_0^1 (\frac{z}{2} + \frac{(1-z)^3 - 1}{6}) dz \\ &= (\frac{z^2}{4} - \frac{(1-z)^4}{24} - \frac{z}{6})|_{z=0}^{z=1} = \frac{1}{4} - \frac{1}{6} + \frac{1}{24} = \frac{1}{8}. \end{aligned}$$

9. There are five other possible orders of integration, so how to judge which of these five to use? Because integrating $e^{x^3}$ with respect to $x$ is impossible as it stands, that suggests leaving the $dx$ integral for last, integrating first with respect to $y$ and then $z$, or vice versa. So let's try the following order: first $dy$, then $dz$ then $dx$. From the original iterated integral, the inequalities satisfied by the three variables are given as

(a): $0 \le z \le 1$, (b): $z \le x \le 1$ and (c): $0 \le y \le x$.

Since $x$ is the last variable to be integrated, we need to find the constant upper and lower limits for it. From (b), we have $x \le 1$, also from (b), we have $x \ge z$, but $z$ is not a constant, so we combine (a) and (b) and obtain $0 \le z \le x$ so $0 \le x$, thus $0 \le x \le 1$.

We then determine the upper and lower limits for $z$, obviously the upper and lower limits for $z$ can be functions of $x$ but can not contain $y$. From (b), we have $z \le x$ so the upper limit for $z$ is $x$, from (a), we have $0 \le z$ so the lower limit for $z$ is 0, thus $0 \le z \le x$.

We finally determine the upper and lower limits for $y$, and they can be functions of both $x$ and $z$. From (c), we have $0 \le y \le x$. So we obtain the following iterated integral:

$$\begin{aligned}
\int_0^1 dz \int_z^1 dx \int_0^x e^{x^3} dy &= \int_0^1 dx \int_0^x dz \int_0^x e^{x^3} dy \\
&= \int_0^1 dx \int_0^x (ye^{x^3})|_{y=0}^{y=x} dz \\
&= \int_0^1 dx \int_0^x xe^{x^3} dz = \int_0^1 (zxe^{x^3})|_{z=0}^{z=x} dx \\
&= \int_0^1 x^2 e^{x^3} dx = \frac{1}{3}(e^{x^3})|_{x=0}^{x=1} = \frac{e-1}{3}.
\end{aligned}$$

Remark : The order of integration first $dz$, then $dy$, last $dx$ works about as easily as the order used above.