The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #5 Answers

1. It's easy to check, following the hint, that $\mathbf{F}$ is conservative. Also the curve $\mathcal{C}$, being the intersection of a paraboloid with an oblique plane, is a closed curve. That's all one needs, the line integral $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = 0$ for any conservative vector field around any closed curve.

2. This question is routine since the curve $\mathcal{C}$ is already parametrized. You should find that the integral reduces to $\int_0^1 (e^t + 2te^t + 2te^t)dt$, and the answer is $e + 3$.

3. Following the hint, notice that the portion $\langle ye^{xy}, xe^{xy}\rangle$ of the given vector field is conservative (a potential function for this would be $\phi = e^{xy}$), so this part integrates to 0 around the closed curve $\mathcal{C}$. Therefore just use the remaining part of the vector field $\langle 0, x\rangle$. As usual parametrize the curve, which is the unit circle, by $\mathbf{r} = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. It is now routine to find that the integral reduces to $\int_0^{2\pi} (\cos(t))^2 = \pi$, where the last integral is worked out using the double angle formula for $(\cos t)^2$.

4. (a) Use $\mathbf{r} = \langle 2\cos u, 2\sin u, v\rangle$. One finds $\|\mathbf{r}_u \times \mathbf{r}_v\| = 2$, so $dS = 2dudv$. The surface area is then given by
$$\int_0^{\pi/2} \int_0^{5-4\cos u - 2\sin u} 2\, dv\, du = 5\pi - 12.$$
(b) Use $\mathbf{r} = \langle v\cos u, v\sin u, 5 - 2v\cos u - v\sin u\rangle$. One finds $dS = \sqrt{6}v$, and finally gets
$$\int_0^{2\pi} \int_0^2 \sqrt{6}\, v\, dv\, du = 4\sqrt{6}\,\pi.$$

5. Here the simple parametrization $\mathbf{r} = \langle x, y, x^2/2\rangle$ works well. One then finds $dS = \sqrt{x^2+1}dydx$. This gives a pretty tough looking integral but fortunately it works out easily with the substitution $u = 1 - x^4$, we don't show all details (the region is the quarter of the unit disk in the 4'th quadrant) :
$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 \frac{1}{2} x^3 \sqrt{1+x^2}\, dy\, dx = \int_0^1 x^3\sqrt{1-x^4}\, dx = \frac{1}{12}.$$

6. Parametrize with $\mathbf{r} = \langle x, y, 2 - x - y\rangle$. Then $dS = \sqrt{3}\, dv\, du$. Integrating over the elliptical disk $x^2 + 2y^2 \le 1$ gives
$$\int_{-1}^1 \int_{-\frac{\sqrt{2}}{2}\sqrt{1-x^2}}^{\frac{\sqrt{2}}{2}\sqrt{1-x^2}} x^2\sqrt{3}\, dy\, dx.$$
Now the substitution $x = r\cos\theta$, $y = \frac{1}{\sqrt{2}}\sin\theta$ will simplify things. By taking the Jacobian determinant we find $dy\, dx = \frac{r}{\sqrt{2}}\, dr\, d\theta$, and finally get
$$\frac{\sqrt{3}}{\sqrt{2}} \int_0^{2\pi} \int_0^1 r^2 (\cos\theta)^2 \cdot r\, dr\, d\theta = \frac{\sqrt{6}}{8}\pi.$$

7. There are three regions (surfaces) and one has to add the flux over each one. The bottom region is the disk $z = 0,\ x^2 + y^2 \leq 4$, with outward unit normal $\mathbf{N} = \langle 0, 0, -1\rangle$. Here $\mathbf{F} = \langle 1, 1, 0\rangle$ (since $z = 0$), so $\mathbf{F} \bullet \mathbf{N} = 0$ and the flux for this part is 0. The top region is similar, with $\mathbf{N} = \langle 0, 0, 1\rangle$ and also $z = 3$. One then gets for this region

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3(x^2 + y^2)^2\, dy\, dx,$$

and this is easily done by converting to polar coordinates

$$3 \int_0^{2\pi} \int_0^2 r^4 \cdot r\, dr\, d\theta = 64\pi.$$

For the cylindrical part, it is the same cylinder as in 4(a) so use the same parametrization. It gives $\mathbf{r}_u \times \mathbf{r}_v = \langle 2\cos u, 2\sin u, 0\rangle$, so $\mathbf{N} dS = \langle 2\cos u, 2\sin u, 0\rangle du\, dv$, giving for the flux on this part

$$\int\int \mathbf{F} \bullet \mathbf{N} dS = 2 \int_0^{2\pi} \int_0^3 (\cos u + \sin u) dv\, du = 2 \cdot 3 \cdot \int_0^{2\pi} (\cos u + \sin u)\, du = 0.$$

Adding the three contributions gives 64π.

Remark : This problem can be done a second way, using the divergence theorem, and it is much easier this way. It will be on Handout 6.

8. Use $\mathbf{r} = \langle x, y, \sqrt{4 - y^2}\rangle$. One finds

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 0, \frac{y}{\sqrt{4 - y^2}}, 1\rangle,$$

$$\int\int \mathbf{F} \bullet \mathbf{N}\, dS = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (\frac{y^2}{\sqrt{4 - y^2}} + x\sqrt{4 - y^2})\, dx\, dy.$$

This hard looking integral actually simplifies lots when done, giving 16/3.

Remark : A good sketch of the surface is helpful to see what the region of integration in the xy-plane should be. It is the semidisk $0 \leq y \leq \sqrt{4 - x^2}$, which should explain where the limits of integration on the double integral come from.