The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #6 Answers

1. This is a straightforward application of Green's theorem. Applying it gives

$$\int_0^1 \int_{3x}^3 (4y^3 - 2x^2y^2)dydx,$$

and there is no problem integrating this to obtain 318/5.

2. This is not a closed curve, and a direct attempt at the line integral will lead to very difficult (impossible) integration. To apply Green's theorem, let's complete the given curve, which we will call $\mathcal{C}_1$, to a simple closed curve by adding the line segment $\mathcal{C}_2$ from $(0,0)$ to $(\pi, 0)$. We then get a simple closed curve $\mathcal{C} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$. Then

$$\int_{\mathcal{C}} = \int_{\mathcal{C}_1} - \int_{\mathcal{C}_2}.$$

Using Green's theorem (noting $\mathcal{C}$ has clockwise orientation),

$$\int_{\mathcal{C}} = -\int_0^\pi \int_0^{\sin x} (2x - 3y^2)dydx = \int_0^\pi ((\sin x)^3 - 2x\sin x)dx,$$

which gives $4/3 - 2\pi$ after a bit of work.

For $\mathcal{C}_2$ use the parametrization $\mathbf{r} = \langle t, 0\rangle$ and one gets

$$\int_0^\pi \sqrt{t}\ dt = (2/3)(\pi)^{3/2}.$$

The final answer is then $4/3 - 2\pi + (2/3)(\pi)^{3/2}$.

3. Directly using the divergence theorem gives

$$\int\int\int_{\mathcal{R}} (12x^2z + 12y^2z + 12z^3)dV,$$

and changing to spherical coordinates gives

$$12\int_0^{2\pi}\int_0^\pi\int_0^R (\rho^2 z)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 0.$$

4. $\nabla \bullet \mathbf{F} = 0 + 0 = 0$, so the answer is 0.

5. In this case Stokes's theorem is used in "reverse." In this case our surface $\mathcal{S}$ is the paraboloid and its boundary $\mathcal{C} = \partial\mathcal{S}$ is the circle of radius 2 in the plane $z = 5$, centred at the origin. Parametrize $\mathcal{C}$ by $\mathbf{r} = \langle 2\cos t, 2\sin t, 5\rangle$. One finds

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} 20(-\sin^2 t + \cos^2 t)dt = 20\int_0^{2\pi} \cos(2t)dt = 0.$$

6. Parametrize by $\mathbf{r}(x,y) = \langle x, y, 1 - x - y/2\rangle$. Applying Stokes's theorem then leads fairly directly to

$$\int_0^1 \int_0^{2x-2} e^x dydx = 2e - 4.$$