Math 381

Assignment 1, Key

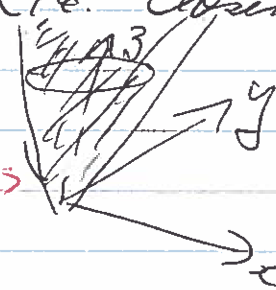
10.1 22 $z \geq \sqrt{x^2+y^2}$ Note $z \geq 0$.

Squaring gives $z^2 \geq x^2+y^2$. Consider $z^2 = x^2+y^2$

This is a surface of revolution about the z-axis.

The cross section in $y=0$ is a pair of lines $z = \pm x$

so $z^2 = x^2+y^2$ ~~is~~ is the two napped cone obtained by rotating $z=x$ about the z-axis. Thus $z \geq \sqrt{x^2+y^2}$ is the set of points inside (i.e. closer to the z axis) the upper half cone.

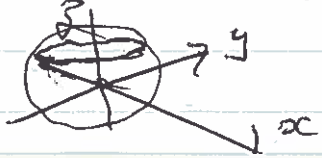


1 - level curves are discs
1 - straight edges
1 - merrar

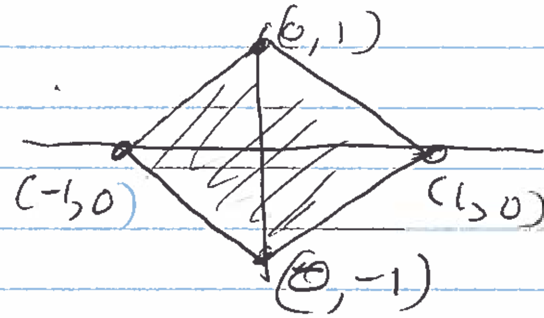
1 - shape is cone
1 - style.

26 $x^2+y^2+z^2=4$
$z=1$.

$x^2+y^2+z^2 = 2^2$ is the sphere of radius 2 centered at the origin. $z=1$ is the horizontal plane through $(0,0,1)$. So the intersection is the circle (of latitude) $x^2+y^2=3$, $z=1$ centred at $(0,0,1)$ in the plane $z=1$



36 $|x| + |y| \leq 1$



Boundary

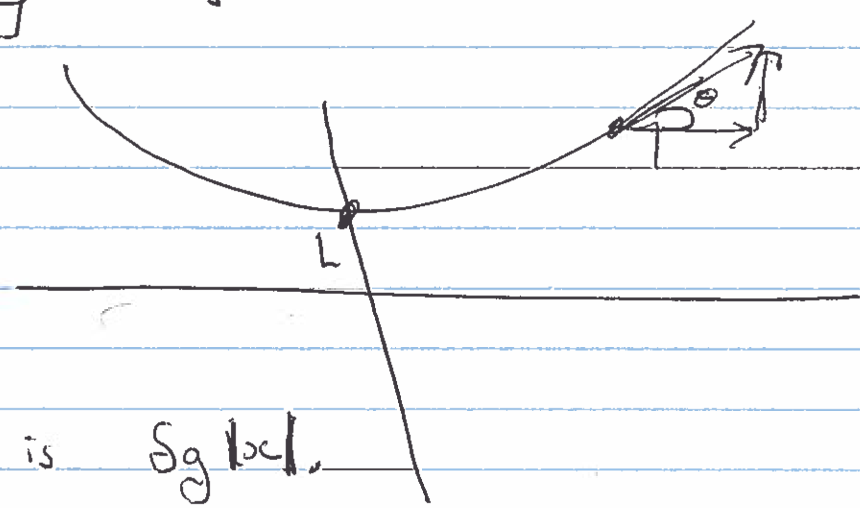
is the union of the four

line segments joining $(0,1)$ $(1,0)$, $(1,0)$,$(0,-1)$, $(0,-1)$, $(-1,0)$ and $(-1,0)$,$(0,1)$.

The interior is the inside of this "diamond" shape

$\{(x,y) : |x|+|y| < 1\}$.

10.2 3



Weight supported at P is $\delta g |x|$.

By symmetry considerations we consider $x \geq 0$. $\frac{dy}{dx} = 0$ at $x=0$ Let the horizontal component of the tension be h (= Tension at $x=0$.)

Resolving the tension T at P: horizontal component $T\cos\theta = h$

Vertical Component $\quad T'\sin\theta = \delta g x$

Dividing we get

$$\frac{h}{\cos\theta}\sin\theta = \delta g x$$

1 3+YLG $\quad h\tan\theta = \delta g x$

So $\quad \frac{dy}{dx} = \frac{\delta g}{h} x$ 1

Let $\frac{\delta g}{h} = a$, a Constant. 1

Integrating we get y 1 $= \frac{a}{2}x^2 + C$

where C is the y-component of L. We may set $C = 0$.

Then $y = \frac{a}{2}x^2$, which is a parabola

32 Suppose that $\underline{u}$, $\underline{v}$ and $\underline{r}$ are position vectors of points U, V and P, respectively that $\underline{u}$ is not parallel to $\underline{v}$ and P lies on the plane containing the origin $(0,0,0)$, U and V,

Then we can project $\underline{v}$ in the direction of $\underline{u}$, obtaining $\underline{v}_p = \frac{\underline{u} \circ \underline{v}}{|u|^2} \underline{u}$, and the component orthogonal to $\underline{u}$ is ~~v_o~~.

$$\underline{v}_o = \underline{v} - \underline{v}_p = \underline{v} - \frac{(\underline{u} \circ \underline{v})}{|u|^2} \underline{u}$$

Similarly $\overrightarrow{OP} = \underline{r}$ can be resolved into

$$\underline{r}_p = \frac{\underline{r} \circ \underline{u}}{|u|^2} \underline{u} \quad \text{and} \quad \underline{r}_o = \underline{r} - \frac{r \circ u}{|u|^2} \underline{u}$$

with $\underline{r}_p$ in the direction of $\underline{u}$, and $\underline{r}_o$ orthogonal to $\underline{u}$.

Then $\underline{r}_p$ and $\underline{v}_p$ are both in the direction of $\underline{u}$ so each is a scalar multiple of the other. WLOG $\underline{r}_p = a \underline{v}_p$

Also the vectors $\underline{v}_o$ and $\underline{r}_o$ both lie on the plane of U, V, P and are perpendicular to $\underline{u}$ so they are parallel. So there's b with $\underline{r}_o = b \underline{v}_o$.

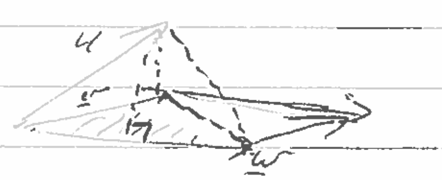
Now
$$\underline{r} = \underline{r}_p + \underline{r}_o = a\underline{v}_p + b\underline{v}_o$$
$$= a\underline{v}_p + b\left(\underline{v} - \frac{\underline{v}\circ\underline{u}}{|u|^2}\underline{u}\right)$$
$$= a\frac{\underline{v}\circ\underline{u}}{|\underline{u}|^2}\underline{u} + b\underline{v} - b\frac{\underline{v}\circ\underline{u}}{|u|^2}\underline{u}$$
$$= (a-b)\frac{\underline{v}\circ\underline{u}}{|u|^2}\underline{u} + b\underline{v}$$

Setting $\lambda = (a-b)\dfrac{\underline{v}\circ\underline{u}}{|u|^2}$, and $\mu = b$

we get $\underline{r} = \lambda\underline{u} + \mu\underline{v}$, as desired.

10.3 14 Volume of tetrahedron spanned by

$\underline{u}, \underline{v}, \underline{w}$ is $\frac{1}{3}Ah$ where A is Area of base



and h is height.

The base of the tetrahedron has area one half the area of the parallelogram spanned by $\underline{v}, \underline{w}$, i.e. numerically $\frac{1}{2}|\underline{v}\times\underline{w}|$.

If we drop a perpendicular from the tip of $\underline{u}$ to the plane of $\underline{v}, \underline{w}$ we obtain the height h as a resultant. Now $\underline{v} \times \underline{w}$ is perpendicular to the plane of $\underline{v}$ and $\underline{w}$ and so parallel to this perpendicular. So h is the length of the projection of $\underline{u}$ in the direction of $\underline{v} \times \underline{w}$, that

is $h = \left| \dfrac{\underline{u} \circ (\underline{v} \times \underline{w})}{|\underline{v} \times \underline{w}|} \right|$

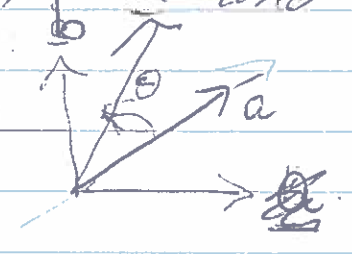
$$\text{So } V = \frac{1}{3} h A = \frac{1}{3} \left| \frac{\underline{u} \circ (\underline{v} \times \underline{w})}{|\underline{v} \times \underline{w}|} \right| \cdot \frac{1}{2} |\underline{v} \times \underline{w}|$$

$$= \frac{1}{6} \left| \underline{u} \circ (\underline{v} \times \underline{w}) \right|$$

as required, and as $|\underline{u} \circ (\underline{v} \times \underline{w})|$ is the volume of the parallelopiped, the result follows.

28 To solve for $\underline{a} \times \underline{X} = \underline{b}$, with $\underline{a}, \underline{b}$ non-zero, note that we must have $\underline{b}$ orthogonal to

Suppose that $\underline{a}$ and $\underline{b}$ are nonzero and orthogonal



By rotations and ~~a~~ we can assume b is in the direction of the z axis and $\underline{a}$ in that of the y axis. ~~Then $\underline{x}$ must be in the (y, z) plane~~

Since $\underline{b}$ is orthogonal to $\underline{x}$!, $\underline{x}$ must be in the x, y plane. For convenience the drawing shows one orientation. Now let θ be an angle swept out in the x, y plane with ~~$\theta \neq \pi$~~ $0 < \theta < \pi$ ~~$\theta \neq k\pi$~~, k integer, and let $\underline{x}$ be the vector in the direction of the ray defined by θ with length $\dfrac{|b|}{|a|\sin\theta}$. Then

$\underline{a} \times \underline{X}$ points in the direction of $\underline{b}$, and has length $\dfrac{|b|}{|a|\sin\theta}\,|a|\sin\theta = |b|$, so solves the equation

The solution is <u>not</u> unique, there are infinitely many in fact.

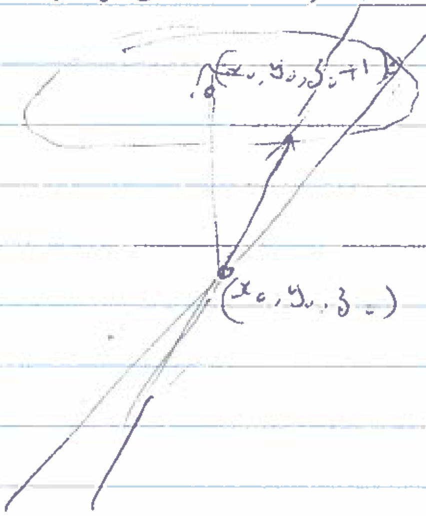
10.4.32

Let λ be a parameter

and $\dfrac{x-x_0}{\sqrt{1-\lambda^2}} = \dfrac{y-y_0}{\lambda} = z-z_0 = s$, say.

Then $(x-x_0,\ y-y_0,\ z-z_0) = (\sqrt{1-\lambda^2},\ \lambda,\ 1)s$

Note $\lambda \in (-1, 1) - \{0\}$, and $\sqrt{1-\lambda^2} > 0$.



Lines have direction vectors to points on circle $x^2+y^2=1$ $z=1$, with $0<x<1$ and $-1<y<1$, $y \neq 0$

10.6 10

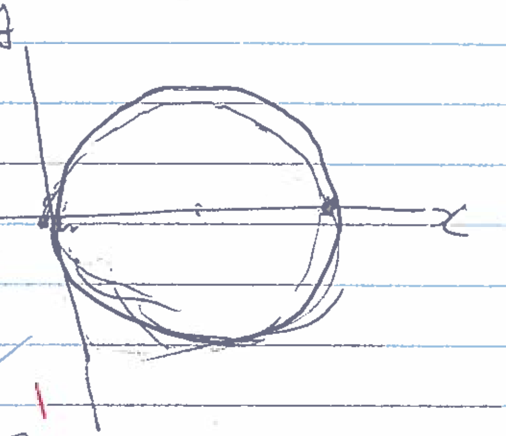
$\rho = z$ in spherical coordinates.

As $z = \rho\cos\varphi$ we must have $\cos\varphi = 1$ so $\varphi = 0$.

This describes the non-neg. z axis, includ. $(0,0,0$

14 $r = 2\cos\theta$

Vertical Cylinder through the curve in the plane $z = 0$ with polar coordinates (r, θ) where



$r = 2\cos\theta$

$x = 2\cos^2\theta = 1 + \cos 2\theta$ 1

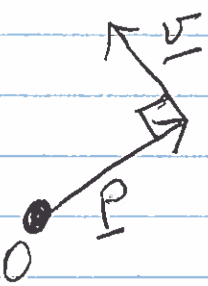
$y = 2\cos\theta\sin\theta = \sin 2\theta$ 1

So $(x-1)^2 + y^2 = \cos^2 2\theta + \sin^2 2\theta = 1$ 1

Circle Centre $(1, 0)$ radius 1.

1 style

11.1 #24



$$\frac{d\underline{P}}{dt} = \underline{v}$$

$$\frac{d}{dt}(\underline{P} \circ \underline{P}) = \frac{d\underline{P}}{dt} \circ \underline{P} + \underline{P} \circ \frac{d\underline{P}}{dt}$$

$$= 2\underline{v} \circ \underline{P} = 0$$

So $\underline{P} \circ \underline{P} = \rho_0^2$ constant.

So $\underline{P}$ moves on the sphere of radius ρ_0 centred at the Origin

11.3 16

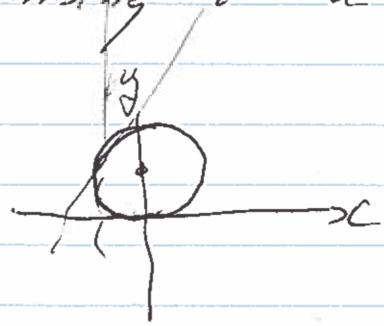
$$x = a\cos t \sin t \qquad y = a\sin^2 t, \ z = bt$$

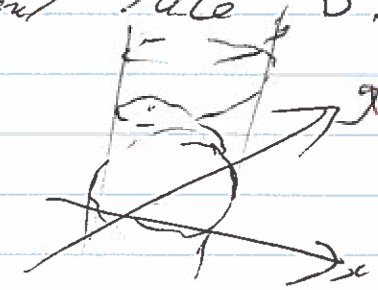
$$\boxed{x = \frac{a}{2}\sin(2t) \qquad y = \frac{a}{2}(1-\cos 2t), \ z = bt}$$

$$\frac{2x}{a} = \sin 2t \qquad \left(1-\frac{2y}{a}\right) = \cos 2t \qquad z = bt$$

$$\left(\frac{2x}{a}\right)^2 + \left(1-\frac{2y}{a}\right)^2 = \sin^2 2t + \cos^2 2t = 1$$

Pt moves on the surface of the circular cylinder centred at $(0, \frac{a}{2})$ of radius $\frac{a}{2}$ climbing at a constant rate b.

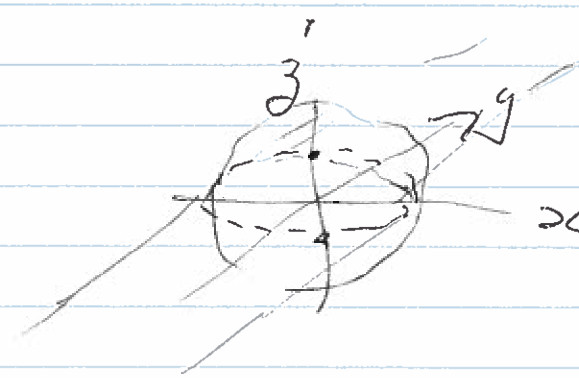




1

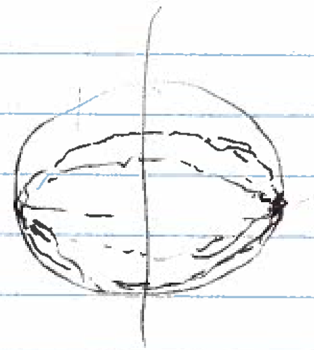
Length $T\sqrt{a^2+b^2}$

Spiral.



18. $\left.\begin{array}{l} x^2+y^2+z^2=1 \\ x^2+2z^2=1 \end{array}\right\}$

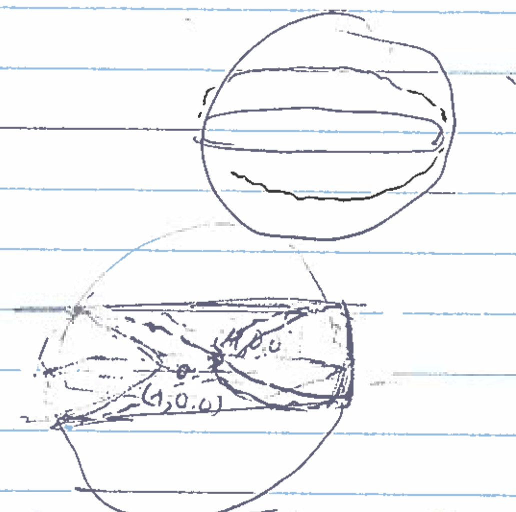
Subtracting gives $y^2 - z^2 = 0$ $\qquad \underline{y = \pm z}$



The curve traces out a pair of paths ~~from~~ on the sphere from $(1,0,0)$ to $(-1,0,0)$, ~~through~~ and back, one on the positive side, one on the negative side $y \geq 0$, $y \leq 0$.

$$P(\varphi) = \left(\cos\varphi, \frac{\sin\varphi}{\sqrt{2}}, \frac{\sin\varphi}{\sqrt{2}}\right)$$

and $\left(\cos\varphi, \frac{-\sin\varphi}{\sqrt{2}}, \frac{\sin\varphi}{\sqrt{2}}\right)$



For one part, with $0 \leq \varphi \leq 2\pi$

$$\frac{dP}{d\varphi} = \left(-\sin\varphi, \frac{\cos\varphi}{\sqrt{2}}, \frac{\cos\varphi}{\sqrt{2}}\right)$$

$$\left|\frac{dP}{d\varphi}\right| = \sqrt{\sin^2\varphi + \frac{\cos^2\varphi}{2} + \frac{\cos^2\varphi}{2}} = 1, \quad s = \int_0^{2\pi} 1\, d\varphi = 2\pi.$$ Total length is twice so 4π

11.4 Suppose $K(s) = K_0$ and $\tau(s) = \tau_0$ are both constants. Consider the circular helix parametrized by $P(s) = a\cos cs\ \underline{i} + a\sin cs\ \underline{j} + bcs\ \underline{k}$

where $c = \frac{1}{\sqrt{a^2+b^2}}$

given in Example 3. There it is shown that $K(s) = ac^2 = \frac{a}{a^2+b^2}$

0.5

Moreover $\tau(s) = -(-bc^2) = \dfrac{b}{a^2+b^2}$. 0.5

Setting $\dfrac{a}{a^2+b^2} = \kappa_0$, $\dfrac{b}{a^2+b^2} = \tau_0$, $\kappa_0 \tau_0 \neq 0$

we can write $a^2+b^2 = \dfrac{a}{\kappa_0} = \dfrac{b}{\tau_0}$

So $a\tau_0 = b\kappa_0$. Taking $a = \kappa_0$, $b = \tau_0$

$$a = \frac{b\kappa_0}{\tau_0}, \quad a^2+b^2 = \left(\frac{b^2\kappa_0^2}{\tau_0^2}\right) + b^2 = b^2\left(\frac{\kappa_0^2+\tau_0^2}{\tau_0^2}\right) = \frac{b}{\tau_0}$$

Giving $b = \dfrac{\tau_0}{\kappa_0^2+\tau_0^2}$, $a = \dfrac{\kappa_0}{\kappa_0^2+\tau_0^2}$. 2

With this choice of a, b the circular helix has the property that $\kappa(s) = \kappa_0$, $\tau(s) = \tau_0$ and we can apply thm 3. The given curve is congruent to a circular helix.

STYLE