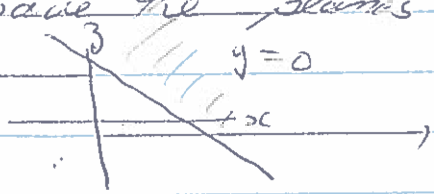
Math 381
Assignment 4 - Solutions

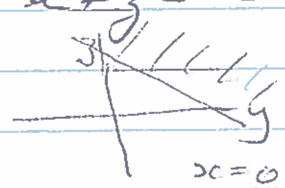
13.2 Problem 18

Let $F(x,y,z) = 2x + 3y + 4z$ subject to

$x \geq 0$, $y \geq 0$, $z \geq 0$, $x+y \geq 2$, $y+z \geq 2$, $x+z \geq 2$.

Since $\nabla F = 2\underline{i} + 3\underline{j} + 4\underline{k}$ the minimum must occur at a vertex which is extremal in the direction $-\nabla F$. So we do not need to be concerned that the region is unbounded. The region lies in the first octant, to the right of the vertical plane through the line $z=0$, $x+y=2$ passing through $(2,0,0)$ and $(0,2,0)$. Note that neither $(2,0,0)$ nor $(0,2,0)$ are vertices of the polyhedron since they fail to satisfy $y+z=2$ and $x+z=2$ respectively. The region lies above the planes $x+z=2$ and $y+z=2$,





There are six boundary planes which give rise to $_6C_3 = 20$ candidate vertices, if one wants to do an exhaustive search. Note that setting any one of $x, y,$ or $z = 0$ yeilds the other two taking the value at least 2 (from $x+y \geq 2$, $y+z \geq 2$ and $x+z \geq 2$) and leads to a contradiction. So $x=0$, $y=0$, $z=0$ are not defining planes for a feasible vertex. This leads us to solve $x+y=2$, $y+z=2$, $x+z=2$ leading to $x=1, y=1, z=1$ and $F(x,y,z) = 9$.

13.3 Problem 10 Consider $F(x,y,z) = x+2y-3z$ over the ellipsoid $\mathcal{E}: x^2+4y^2+9z^2 \leq 108$. Because $\mathcal{E}$ is closed and bounded F achieves a minimum and a maximum but, since $\nabla F = (1,2,-3) \neq \underline{0}$ this must occur on the boundary $G(x,y,z) = x^2+4y^2+9z^2-108 = 0$.

So we introduce a LaGrange multiplier λ and set

$$\nabla F = \lambda \nabla G$$

$$(1, 2, -3) = \lambda(2x, 8y, 18z)$$

$$= \mu(x, 4y, 9z) \text{ where}$$

$\mu = 2\lambda$.

$$1 = \mu x$$
$$2 = 4\mu y$$
$$-3 = 9\mu z$$
and $x^2 + y^2 + z^2 = 108$.

Thus $\mu, x, y, z \neq 0$ and $x = \frac{1}{\mu}$, $y = \frac{1}{2\mu}$, $z = \frac{-1}{3\mu}$

so $\frac{1}{\mu^2} + \frac{4}{4\mu^2} + \frac{9}{9\mu^2} = 108$ and $\frac{3}{\mu^2} = 108$

$\mu^2 = \frac{1}{36}$, $\mu = \pm\frac{1}{6}$. This gives

$(x, y, z) = (6, 12, -18)$ and $(x, y, z) = (-6, -12, 18)$

$F(6, 12, -18) = 6 + 24 - 54 = -24$, min

$F(-6, -12, 18) = -6 - 24 + 54 = 24$, max.

B, 3 Problem 12 Let $f(x, y, z) = x^2 + y^2 + z^2$

subject to $G(x,y,z) = x^2 + y^2 - z^2 = 0$ and

$H(x,y,z) = x - 2z - 3 = 0.$

Introducing LaGrange Multipliers λ, μ

and setting $\nabla f = \lambda \nabla G + \mu \nabla H$

we obtain

$$(2x, 2y, 2z) = \lambda(2x, 2y, -2z) + \mu(1, 0, -2)$$

and $\quad x^2 + y^2 - z^2 = 0, \quad x - 2z - 3 = 0$

$$2x = 2\lambda x + \mu$$
$$2y = 2\lambda y$$
$$2z = -2\lambda z - 2\mu$$

$$x^2 + y^2 = z^2$$
$$x - 2z = 3$$

Case 1 $\quad y = 0, \quad x = \pm z \quad$ If $x = z$, $x = z = -3$,

$$\left.\begin{array}{l} -6 = -6\lambda + \mu \\ -6 = 6\lambda - 2\mu \end{array}\right\} \mu = 12, \lambda = 3$$

Giving $(x,y,z) = (-3, 0, -3)$

If $x = -z \qquad x = 1, \; z = -1, \; \mu = \frac{4}{3}, \; \lambda = \frac{1}{3}$

giving $(x,y,z) = (1, 0, -1)$

Case 2 $y \neq 0$ Then $\lambda = 1$ $\mu = 0$, $z = 0$

$x = 3$, which is impossible for $x^2 + y^2 = z^2$.

Now $f(-3, 0, -3) = 9 + 0 + 9 = 18$ max

$f(1, 0, -1) = 1 + 0 + 1 = 2$ min.

13.3 Problem 22 Let $f(x, y, z) = xy + z^2$

on the ball $x^2 + y^2 + z^2 \leq 1$.

For interior points set $\nabla f = (y, x, 2z) = \underline{0}$

giving the origin $(0, 0, 0)$.

For the boundary $G(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

and we set $\nabla f = \lambda \nabla G$, $G = 0$

giving

$$y = 2\lambda x$$
$$x = 2\lambda y$$
$$2z = 2\lambda z$$
$$x^2 + y^2 + z^2 = 1$$

If $z \neq 0$, $\lambda = 1$, $x = 2y$, $2x = y \Rightarrow x = y = 0$,

$z = \pm 1$. This gives $(0, 0, 1)$ and $(0, 0, -1)$.

If $z=0$ we cannot have $x=0$ or $y=0$ (or both are given $x^2+y^2+z^2=0$) $y=2\lambda x=2\lambda\, 2\lambda y$ gives $y=4\lambda^2 y$ so $\lambda=\pm\frac{1}{2}$. This gives $x=y$ or $x=-y$. From $x^2+y^2+0^2=1$ we get

$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, $\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, $\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right)$

and $\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right)$.

Calculating $f(x,y,z)=xy+z^2$

$$f(0,0,0)=0$$
$$f(0,0,1)=1=f(0,0,-1)$$
$$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)=\frac{1}{2}=f\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right)$$
$$f\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right)=\frac{-1}{2}=f\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

So the minimum value is $\frac{-1}{2}$ and the maximum is 1.

15.4 Problem 12 Let $y=p+qx$. Find the selection of p & q that give the best least squares fit to $(1,0.11)$, $(2,1.62)$, $(3,4.07)$, $(4,7.55)$, $(6,17.63)$

and $(7, 24.20)$

$$\text{minimize } E = \sum (y_i - p x_i - q)^2$$

$$\frac{\partial E}{\partial p} = -2 \sum (y_i - p x_i - q) x_i$$

$$\frac{\partial E}{\partial q} = -2 \sum (y_i - p x_i - q) \cdot 1$$

Setting $\frac{\partial E}{\partial p} = 0 = \frac{\partial E}{\partial q}$ we get

$$p \sum x_i^2 + q \sum x_i = \sum y_i x_i$$

$$p \sum x_i + q \sum 1 = \sum y_i$$

Now $\sum 1 = 6$, $\sum x_i = 1+2+3+4+6+7 = 23$

$$\sum x_i y_i = 1 \cdot 0.11 + 2 \cdot 1.62 + 3 \cdot 4.07 + 4 \cdot 7.55 +$$

$$6 \cdot 17.63 + 7 \cdot 24.20 = 320.94$$

$$\sum y_i = 0.11 + 1.62 + 4.07 + 7.55 + 17.63 + 24.20$$

$= 55.18$ $\quad \sum x_i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 7^2 = 115$

So we have

$$115p + 23q = 320.94$$

$$23p + 6q = 24.20$$

$p = \frac{1369.04}{161} = 8.503354$ and $q = \frac{-2759.72}{161 \times 6} = -28.562856$ ...

The predicted value for $x = 5$ is $5 \cdot \frac{1369.04}{161} - \frac{2759.72}{966}$

13.4 Problem 18. Consider $F(a,b,c) = \int_0^1 (x^3 - ax^2 - bx - c)^2 dx$

$$\frac{\partial F}{\partial a} = \int_0^1 2(x^3 - ax^2 - bx - c)(-x^2)\, dx$$

$$\frac{\partial F}{\partial b} = \int_0^1 2(x^3 - ax^2 - bx - c)(-x)\, dx$$

$$\frac{\partial F}{\partial c} = \int_0^1 (x^3 - ax^2 - bx - c)(-1)\, dx$$

Setting all three to 0 gives

$$a\int_0^1 x^4 dx + b\int_0^1 x^3 dx + c\int_0^1 x^2 dx = \int_0^1 x^5 dx$$

$$a\int_0^1 x^3 dx + b\int_0^1 x^2 dx + c\int_0^1 x\, dx = \int_0^1 x^4 dx$$

$$a\int_0^1 x^2 dx + b\int_0^1 x\, dx + c\int_0^1 1\, dx = \int_0^1 x^3 dx$$

So
$$\begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{5} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{5} \\ \frac{1}{4} \end{bmatrix}$$

13.5 Problem 6 Let $F(x) = \int_0^\infty e^{-xt} \frac{\sin t}{t}\, dt$

Note that $\frac{\partial}{\partial x}\left(e^{-xt}\frac{\sin t}{t}\right) = e^{-xt}\sin t$

So $\int_0^\infty e^{-xt}\sin t\, dt = \frac{d}{dx}\int_0^\infty -e^{-xt}\frac{\sin t}{t}\, dt$

By Problem 15 $\int_0^\infty e^{-xt}\sin t\, dt = \frac{1}{1+x^2}$ for $x>0$

and so $\frac{d}{dx}\int_0^\infty e^{-xt}\frac{\sin t}{t}\, dt = -\frac{1}{1+x^2}$

and $F(x) = \int_0^\infty e^{-xt}\frac{\sin t}{t}\, dt = \int -\frac{1}{1+x^2}\, dx$

$= -\operatorname{Arc\,tan} x + C$

Now $\lim_{t\to 0} \frac{\sin t}{t} = 1$ so $\left|e^{-xt}\frac{\sin t}{t}\right| \le K e^{-xt}$

And $\int_0^N e^{-xt}\, dt = -\frac{e^{-xt}}{x}\Big|_0^N = -\frac{e^{-xN}}{x} + \frac{e^0}{x} \to \frac{1}{x}$

as $N \to \infty$. So $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{1}{x} = 0.$

Now $\lim_{x\to\infty} \text{Arctan}\, x = \frac{\pi}{2}$, So $0 = -\frac{\pi}{2} + C$

and $C = \frac{\pi}{2}$.

So $F(x) = \frac{\pi}{2} - \text{Arctan}\, x$.

Now $\int_0^\infty \frac{\sin t}{t}\,dt = \lim_{x\to 0} \int_0^\infty e^{-xt} \frac{\sin t}{t}\,dt = \lim_{x\to 0} \frac{\pi}{2} - \text{Arctan}\, x$

$= \frac{\pi}{2}$.

13.5 Problem 20 Consider the family of curves parametrized by c, for fixed k given by

$$x^2 + (y-c)^2 = kc^2$$

Let $f(x,y,c) = x^2 + (y-c)^2 - kc^2$.

$$\frac{\partial f}{\partial c} = -2(y-c) - 2kc = 2(1-k)c - 2y$$

Setting $f = 0$, $\frac{\partial f}{\partial c} = 0$

$\frac{\partial f}{\partial c} = 0$ gives $c = \frac{y}{1-k}$, $k \neq 1$

$$x^2 + ((1-k)-1)^2 c^2 = kc^2$$

$x^2 + k^2 c^2 = kc^2$ For $k \neq 0, 1$ $\frac{x^2}{k(1-k)} = c^2 = \frac{y^2}{(1-k)^2}$ $y = \pm \frac{1-k}{k} x$

$\pm k y = (1-k)x$ So for $k \neq 0, 1$ the envelope is a pair of lines.

14.1 Problem 8 Let D be the disk $x^2+y^2 \leq 25$ and P the partition of $-5 \leq x \leq 5$, $-5 \leq y \leq 5$ into 100 1×1 unit subboxes. Let $f(x,y) = \begin{cases} 1 & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$

Let (x_{ij}^*, y_{ij}^*) be the corner of the $(i,j)^{th}$ square furthest from the origin. We have $3^2+4^2=5^2$ so there are 5 squares in the bottom left corner first column, 2 in the second column and one each in the third fourth and fifth where the furthest corner lies outside the circle $(5^2+1^2 > 5^2)$, for a total of 9 in each quadrant, and 36 in total, where $f(x_{ij}^*, y_{ij}^*) = 0$. So the Riemann sum is $100-36=64$.

14.2 Problem 2

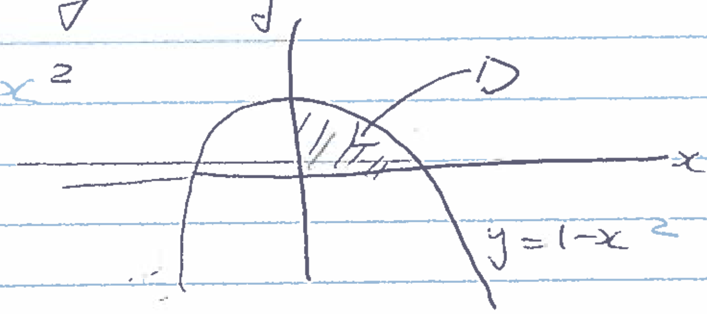
$$\iint_R (xy+y^2)\,dA = \int_0^1\int_0^y (xy+y^2)\,dx\,dy = I$$

~~where R is the triangle~~ with vertices $(0,0)$ $(0,1)$, $(1,1)$

$$I = \int_0^1 \left.\frac{x^2y}{2}+xy^2\right|_{x=0}^{x=y} dy = \int_0^1\left(\frac{y^3}{2}+y^3\right)dy$$

$$= \int_0^1 \frac{3}{2}y^3\,dy = \left.\frac{3}{8}y^4\right|_0^1 = \frac{3}{8}-0 = \frac{3}{8}.$$

14.2 Problem 10 Let D be the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1-x^2$



$$\iint_D x\cos y\,dA$$

$$= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1-x^2} x\cos y\,dy\right)dx = \int_{x=0}^{x=1} \left. x\sin y\right|_{y=0}^{y=1-x^2} dx$$

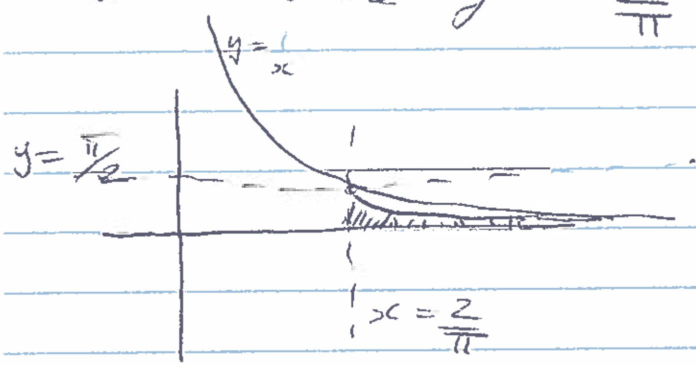
$$= \int_{x=0}^{x=1}\left(x\sin(1-x^2)-0\right)dx = \left.\frac{+1}{2}\cos(1-x^2)\right|_{x=0}^{x=1}$$

$$= \frac{1}{2}\cos(1-1^2) - \frac{1}{2}\cos(1-0^2)$$

$$= \frac{1}{2}(\cos 0 - \cos 1) = \frac{1}{2}(1-\cos 1).$$

14.3 Problem 12 Consider $\iint_R \frac{1}{x}\sin\frac{1}{x}\,dA$

where R is the region $\frac{2}{\pi} < x < \infty$, $0 \le y \le \frac{1}{x}$



Note $0 < \sin\frac{1}{x} \le 1$ on $[\frac{2}{\pi}, \infty)$.

We look at $\lim_{N\to\infty} \int_{x=\frac{2}{\pi}}^{N} \left(\int_{y=0}^{y=\frac{1}{x}} \frac{1}{x}\sin\frac{1}{x}\,dy \right) dx$

$$= \lim_{N\to\infty} \int_{x=\frac{2}{\pi}}^{N} \left. \frac{y\sin\frac{1}{x}}{x} \right|_{y=0}^{y=\frac{1}{x}}$$

$$= \lim_{N\to\infty} \int_{x=\frac{2}{\pi}}^{N} \frac{1}{x^2}\sin\frac{1}{x} \le \lim_{N\to\infty} \int_{x=\frac{2}{\pi}}^{N} \frac{1}{x^2}\,dx$$

$$= \lim_{N\to\infty} \left. -\frac{1}{x} \right|_{x=\frac{2}{\pi}}^{N} = \lim_{N\to\infty} \frac{2}{\pi} - \frac{1}{N} = \frac{2}{\pi}$$

So $\iint_R \frac{1}{x} \sin \frac{1}{x} \, dA$ converges.

Also

$$\lim_{N\to\infty} \int_{x=\frac{2}{\pi}}^{x=N} \frac{1}{x^2} \sin \frac{1}{x} = \lim_{N\to\infty} \cos \frac{1}{x} \Big|_{x=\frac{2}{\pi}}^{x=N}$$

$$= \lim_{N\to\infty} \left(\cos \frac{1}{N} - \cos \frac{\pi}{2} \right)$$

$$= \cos 0 = 1$$

So $\iint_R \frac{1}{x} \sin \frac{1}{x} \, dA = 1$.

14.4 Problem 2 Let D be the disk

$x^2 + y^2 \le a^2$ with $a > 0$

$$\iint_D \sqrt{x^2+y^2} \, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r \; r \, dr \, d\theta = \left(\int_0^{2\pi} 1 \, d\theta \right) \left(\int_{r=0}^{a} r^2 \, dr \right)$$

$$= 2\pi \cdot \frac{a^3}{3} = \frac{2\pi a^3}{3}.$$

14.4 Problem 8 Let Q be the quarter disk given by

$x \ge 0$, $y \ge 0$, $x^2 + y^2 \le a^2$, with $a > 0$.

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$$\iint_Q (x+y)\,dA = \int_{\theta=0}^{y=\pi/2}\int_{r=0}^{a} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \left(\int_{\theta=0}^{\theta=\pi/2} (\cos\theta + \sin\theta)\,d\theta\right)\left(\int_{r=0}^{r=a} r^2\,dr\right)$$

$$= (\sin\theta - \cos\theta)\Big|_{\theta=0}^{\theta=\pi/2} \cdot \frac{a^3}{3}$$

$$= (1-(-1))\,\frac{a^3}{3} = \frac{2a^3}{3}$$