UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 381 — L01 FALL 2009

TERM TEST I
[Friday, October 9, 2009 (in class)]

This is a closed book examination for which you have 50 minutes for completion. Try to give clear, succinct, well-presented answers to as many problems as you can. You can cite relevant theorems where appropriate.

Student ID: KEY

[Marks]

[15] 1. Show that if the position and velocity vectors of a point moving in $\mathbb{R}^3$ are always orthogonal then the point is moving on a path lying on a sphere.

Let the position vector from the origin $(0,0,0)$ to the point at time t be $\underline{r}(t) = (x(t), y(t), z(t))$.

Then the velocity vector $\underline{v}(t) = \underline{r}'(t) = (x'(t), y'(t), z'(t))$

Since $\underline{r}$ and $\underline{v}$ are orthogonal $\underline{r}(t) \circ \underline{v}(t) = 0$.

Consider $\left(\sqrt{x^2+y^2+z^2}\right)^2 = \underline{r}(t) \circ \underline{r}(t)$.

Differentiating $\underline{r}'(t) \circ \underline{r}(t) + \underline{r}(t) \circ \underline{r}'(t) = 2\underline{r}(t) \circ \underline{v}(t)$
$= 0$

So $\underline{r}(t) \circ \underline{r}(t)$ is constant, say k^2; $k > 0$

Thus the path lies on the sphere $x^2+y^2+z^2 = k^2$.

[15] 2. Let $r(s) = (x(s), y(s), z(s))$ be given by $x(s) = a\cos cs$, $y(s) = a\sin cs$, $z(s) = bs$, where $c = \frac{1}{\sqrt{a^2+b^2}}$. Find the unit tangent $\underline{T}(s)$, normal $\underline{N}(s)$ and Binormal $\underline{B}(s)$ as a function of s.

There was a misprint here, $z(s) = bcs$ was intended, so this isn't actually parameterized with respect to arc length.

$$\underline{r}'(s) = (-ac\sin cs,\ ac\cos cs,\ b), \text{ and}$$

$$|\underline{r}'(s)| = \sqrt{a^2c^2+b^2} \quad \text{So arc length } u = \sqrt{a^2c^2+b^2}\, s$$

and $$\underline{T} = \underline{r}'(u) = \left(\frac{-ac}{\sqrt{a^2c^2+b^2}}\sin\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), \frac{ac}{\sqrt{a^2c^2+b^2}}\cos\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), \frac{b}{\sqrt{a^2c^2+b^2}}\right)$$

and $$\frac{dT}{du} = \left(\frac{-ac^2}{a^2c^2+b^2}\cos\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), \frac{-ac^2}{a^2c^2+b^2}\sin\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), 0\right)$$

and $$\kappa(u) = \left|\frac{dT}{du}\right| = \frac{ac^2}{a^2c^2+b^2}, \text{ so}$$

$$\underline{N}(u) = \left(-\cos\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), -\sin\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), 0\right)$$

Thus $$B(u) = T(u) \times N(u)$$

$$= \begin{vmatrix} i & j & k \\ \frac{-ac}{\sqrt{a^2c^2+b^2}}\sin\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right) & \frac{ac}{\sqrt{a^2c^2+b^2}}\cos\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right) & \frac{b}{\sqrt{a^2c^2+b^2}} \\ -\cos\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right) & -\sin\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right) & 0 \end{vmatrix}$$

$$= \left(\frac{b}{\sqrt{a^2c^2+b^2}}\sin\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), \frac{b}{\sqrt{a^2c^2+b^2}}\cos\left(\frac{cu}{\sqrt{a^2c^2+b^2}}\right), \frac{ac}{\sqrt{a^2c^2+b^2}}\right)$$

In terms of s, $$T(s) = \left(\frac{-ac}{\sqrt{a^2c^2+b^2}}\sin cs, \frac{ac}{\sqrt{a^2c^2+b^2}}\cos cs, \frac{b}{\sqrt{a^2c^2+b^2}}\right)$$

$$N(s) = (-\cos cs, -\sin cs, 0) \qquad B(s) = \left(\frac{b}{\sqrt{a^2c^2+b^2}}\sin cs, \frac{b}{\sqrt{a^2c^2+b^2}}\cos cs, \frac{ac}{\sqrt{a^2c^2+b^2}}\right)$$

[15] **3.** **(i) Describe the surface $z = \frac{x^2}{4} + \frac{y^2}{9}$.**

(ii) Find an equation of the line orthogonal to the surface at the point $(2, -3, 2)$.

(iii) Find an equation of the plane tangent to the surface at $(-2, 3, 2)$.

(i) For $z>0$ the cross section is the ellipse $1 = \frac{x^2}{(2\sqrt{z})^2} + \frac{y^2}{(3\sqrt{z})^2}$.

For $z=0$ there is a single point $(0,0,0)$

For $z<0$ the cross section is empty.

Cross sections $x = \text{constant}$ and $y = \text{constant}$ are parabolas opening upward. This is an elliptic paraboloid.

(ii) A normal at (x,y) is given by $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right) = \left(\frac{x}{2}, \frac{2y}{9}, -1\right)$

At $(2,-3,2)$ this gives $\left(1, \frac{-2}{3}, -1\right)$ and an equation of the line in vector form is

$$(x,y,z) = (2,-3,2) + \left(1, \frac{-2}{3}, -1\right)t$$

or

$$x = 2+t, \quad y = -3 - \frac{2}{3}t, \quad z = 2 - t.$$

(iii) A normal is $\left(-1, \frac{2}{3}, -1\right)$ setting $x=-2$ and $y=3$ in the normal equation from (ii)

So an equation of the plane is

$$z = 2 - (x+2) + \frac{2}{3}(y-3)$$

[15] 4. (i) Define "$z = f(x, y)$ is differentiable at (a, b)".

(ii) Suppose $x = x(t)$, $y = y(t)$ define a path in the domain of f and set $g(t) = f(x(t), y(t))$. State the chain rule for $g'(t)$.

(iii) Suppose $x = r\cos\theta$, $y = r\sin\theta$. Let $h(r,\theta) = f(x(r,\theta), y(r,\theta))$. Find $\left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\right)$.

(i) ~~If~~ The first partial derivatives exist and

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - h f_1(a,b) - k f_2(a,b)}{\sqrt{h^2+k^2}} = 0$$

then f is sat to be differentiable at (a,b)

(ii) $g(t) = f(x(t), y(t))$

then $g'(t) = f_1(x(t), y(t)) \cdot \frac{dx}{dt} + f_2(x(t), y(t)) \frac{dy}{dt}$

(iii) $$\left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta, -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta\right)$$

[20] 5. (i) Define "f is harmonic" in a region S.

(ii) Suppose that $u(x,y)$ and $v(x,y)$ have continuous second partial derivatives and satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$. Show both u, and v are harmonic.

(i) f is harmonic in S if it has continuous second partial derivatives and $f_{11} + f_{22} = 0$ on S.

(ii) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x}$$

$$= -\frac{\partial^2 v}{\partial y \partial x}$$

Since $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

and $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ $\quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{+\partial^2 u}{\partial x \partial y}$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

So u and v are both harmonic.

[20] **6. Let $z = e^{x+y}\cos xy$. Define the gradiant vector, and determine the direction in which z is increasing most rapidly at $\left(\frac{\pi}{2}, \frac{1}{2}\right)$, justifying your answer.**

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ where $f(x,y) = e^{x+y}\cos xy$.

$\nabla f = \left(-y e^{x+y}\sin xy,\ -x e^{x+y}\sin xy\right)$

$\nabla f_{\left(\frac{\pi}{2},\frac{1}{2}\right)} = \left(-\frac{1}{2} e^{\frac{\pi+1}{2}}\sin\frac{\pi}{4},\ -\frac{\pi}{2} e^{\frac{\pi+1}{2}}\sin\frac{\pi}{4}\right)$

The direction of maximum increase is that vector since, by the chain rule, the directional derivative in direction $\underline{u}$

$D_u f = \nabla f \circ \underline{u} = |\nabla f||\underline{u}|\cos\theta$

and for $|u| = 1$ this is maximized when $\cos\theta = 1$, i.e. $\theta = 0$.

Name:	Student ID:	Marks: