PMAT 315
SOLUTIONS TO ASSIGNMENT 3
WINTER 2005

1. Page 161, #24. *If $K \lhd G$ and $K \cap G' = \{1\}$, show that $K \subseteq Z(G)$ and that $Z[G/K] = Z(G)/K$.*

   Solution. Let $k \in K$, $g \in G$. Then $gkg^{-1}k^{-1} \in G'$ and $(gkg^{-1})k^{-1} \in gKg^{-1} \cdot K = K^2 = K$. Hence $gkg^{-1}k^{-1} \in G' \cap K = \{1\}$. This holds for all $g \in G$, so $k \in Z(G)$. Now $Z(G)/K \subseteq Z(G/K)$ because each coset Kz, $z \in Z(G)$, commutes with each Kg. If $aK \in Z(G/K)$ then $aga^{-1}g^{-1} \in G' \cap K = \{1\}$. This means $ag = ga$, so $a \in Z(G)$.

2. Page 167, #21. *Show that $\mathbb{C}^*/\mathbb{C}^0 \cong R^+$ where $\mathbb{C}^0 = \{z \in C \mid |z| = 1\}$ is the circle group.*

   Solution. Define $\alpha : \mathbb{C}^* \to \mathbb{R}^+$ by $\alpha(z) = |z|$ ($|z| > 0$ because $z \neq 0$). Then α is a homomorphism because $|zw| = |z|\ |w|$, and $\ker \alpha = \{z \mid |z| = 1\} = \mathbb{C}^\circ$. Thus $\mathbb{C}^*/\mathbb{C}^\circ \cong \alpha(\mathbb{C}^*) \cong \mathbb{R}^+$ by the isomorphism theorem.

3. Page 201, #26. *In each case show that $ab = 1$ in R implies that $ba = 1$.*

   (a) *R is finite.* [Hint: *If $R = \{r_1, r_2, \cdots, r_n\}$ show that $\{br_1, br_2, \cdots, br_n\} = R$.*]

   (b) *Every idempotent in R is central.*

   Solution.

   (a) If $ab = 1$ and $R = \{r_1, r_2, \ldots, r_n\}$ then $br = bs \Rightarrow r = s$ in R, so $\{br_1, br_2, \ldots, br_n\}$ has n elements. Hence $\{br_1, br_2, \ldots, br_n\} = R$. In particular $bc = 1$ for some c. Then $a = a(bc) = (ab)c = c$ so $1 = bc = ba$.

   (b) If $ab = 1$ then $e = ba$ is an idempotent. Thus e is central so $1 = abab = aeb = eab = e1 = e$.

4. Page 212, #26. *Call a ring R "tidy" if every nonzero element is either a unit or a zero divisor.*

   (a) *Describe the "tidy" domains.*

   (b) *Show that every finite ring is "tidy".*

   (c) *Show that every boolean ring (Exercise 34 Section 3.1) is "tidy".*

   (d) *Show that $F(X, \mathbb{R})$ is "tidy" (Example 4, Section 3.1).*

   (e) *If R and S are both "tidy", show that $R \times S$ is "tidy".*

   Solution. (a) They are division rings because there are no zero divisors.

   (b) Let $R = \{r_1, r_2, \ldots, r_n\}$ be a finite ring. If $a \in R$ is not a zero divisor, then $ab = 0$ implies $b = 0$ and $ba = 0$ implies $b = 0$. Hence $aR = \{ar_1, \ldots, ar_n\} = R$ and $R = Ra$ exactly as in Problem 3. Thus $ab = 1 = ca$ for $b, c \in R$. But then $b = c$ is a^{-1} and a is a unit.

   (c) If $0 \neq r \in R$ then $r^2 = r$ so $r(1 - r) = 0$. Hence, if r is not a zero divisor, then $r = 1$ is a unit.

   (d) Let $0 \neq f : X \to \mathbb{R}$ be a function in $F(X, \mathbb{R})$. If $f(x) \neq 0$ for every $x \in X$, define $g(x) = f(x)^{-1}$ for all $x \in X$. Then $fg = 1 = gf$ in $F(X, \mathbb{R})$, so f is a unit. On the other hand, if $f(x_0) = 0$, define $g : X \to \mathbb{R}$ by $g(x) = \begin{cases} 0 & \text{if} \quad x \neq x_0 \\ 1 & \text{if} \quad x = x_0 \end{cases}$. Then $g \neq 0$ (because $|x| \geq 2$) and $fg = 0$. So f is a zero divisor.

   (e) Let $(0, 0) \neq (s, t) \in R$. If $s^{-1} \in S$ and $t^{-1} \in T$ exist, then $(s, t)^{-1} = (s^{-1}, t^{-1})$ in R. Otherwise let (say) $ss_1 = 0$, $s_1 \neq 0$. Then $(s, t)(s_1, 0) = (0, 0)$ in R so (s, t) is a zero divisor.

5. Page 224, #34. *Let R be a commutative ring and let $N(R) = \{a \in R \mid a \text{ is nilpotent}\}$—the* **nil radical** *of R.*

   (a) *Show that $N(R)$ is an ideal of R.* [Hint: *Theorem 4 Section 3.1.*]

   (b) *Show that $N[R/N(R)] = 0$.*

   (c) *Show that $N(R)$ need not be an ideal if R is not commutative.*

   (d) *Show that $N(R)$ is contained in the intersection of all prime ideals of R. (In fact it can be shown that this is equality.)*

   Solution.

(a) If $a^n = 0$ then $(ra)^n = r^n a^n = 0$ for all r. If also $b^m = 0$ consider $(a+b)^{n+m} = \sum_{k=0}^{n+m} \left(\begin{array}{c} n+m \\ k \end{array} \right) a^k b^{n+m-k}$. If $k \geq n$ then $a^n = 0$; if $n + m - k \geq m$ (i.e. $k \leq n$) then $b^{n+m-k} = 0$. So every term in the sum is zero; that is $(a+b)^{n+m} = 0$. Thus $N(R)$ is an ideal.

(b) Write $N = N(R)$. If $r + N \in N(R/N)$ then $(r+N)^n = N$. Then $r^n \in N$ so $(r^n)^m = 0$. Hence $r \in N$, so $r + N = N$ in R/N.

(c) Let $R = M_2(\mathbb{Z}_2)$. Then $a = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right]$ and $b = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right]$ are nilpotent, but $a + b = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$ is not $\left((a+b)^2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \right)$.

(d) If $a^n = 0$ and P is any prime ideal; then $a \cdot a^{n-1} \in P$ so $a \in P$ or $a^{n-1} \in P$. By induction, $a \in P$; so $a \in \cap\{P \mid P \text{ prime ideal}\}$. Thus $N(R) \subseteq \cap\{P \mid \text{is a prime ideal}\}$.