PMAT 315
SOLUTIONS TO ASSIGNMENT 2
WINTER 2005

1. Page 104, #18. *If a and b are numbers, define $\tau_{a,b} : \mathbb{R} \to \mathbb{R}$ by $\tau_{a,b}(x) = ax + b$ for all $x \in \mathbb{R}$. Show that $G = \{\tau_{a,b} \mid a, b \in \mathbb{R},\, a \neq 0\}$ is a subgroup of $S_{\mathbb{R}}$.*

   Solution. We have $1_{\mathbb{R}} = \tau_{1,0} \in G$. Let $\tau_{a,b}$ and $\tau_{a',b'}$ be elements of G. Then $(\tau_{a,b} \cdot \tau_{a',b'})(x) = \tau_{a,b}[\tau_{a',b'}(x)] = \tau_{a,b}(a'x + b') = a(a'x + b') + b = aa'x + (ab' + b)$. Since $aa' \neq 0$, this shows $\tau_{a,b}\tau_{a',b'} = \tau_{aa',ab'+b} \in G$. Finally $(\tau_{a,b})^{-1} = \tau_{a^{-1},-a^{-1}b} \in G$ as the reader can verify. So G is a subgroup of $S_{\mathbb{R}}$ by Theorem 1.

2. Page 113, #26(a). Let $|g| = m$ and $|h| = n$ in a group G where m and n are relatively prime. If $gh = hg$, show that $|gh| = mn$. Is $|gh| = lmc(m, n)$ in general?

   Solution. Write $|gh| = d$. Since $gh = hg$, we have $(gh)^{mn} = g^{mn}h^{mn} = (g^m)^n(h^m)^n = 1$. This means $d \mid mn$. To prove $mn \mid d$, it suffices to show $m \mid d$ and $n \mid d$ (by Theorem 5 §1.2 because $\gcd(m, n) = 1$). This in turn follows if we can show $g^d = 1$ and $h^d = 1$. We have $1 = (gh)^d = g^dh^d$, so $g^d = h^{-d} \in \langle g\rangle \cap \langle h\rangle$. But $\langle g\rangle \cap \langle h\rangle = \{1\}$ because $\gcd(m, n) = 1$. Thus $g^d = 1$ and $h^{-d} = 1$, as required. If $\gcd(m, n) \neq 1$, nothing can be said (for example $h = g^{-1}$).

3. Page 126, #28. *Show that $\mathbb{R}^+ \times \mathbb{C}^0 \cong \mathbb{C}^*$ where $\mathbb{C}^0 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the circle group.*

   Solution. Define $\sigma : \mathbb{R}^+ \times \mathbb{C}^\circ \to \mathbb{C}^*$ by $\sigma(r, z) = rz$. Then σ is one-to-one because $\sigma(r, z) = \sigma(r_1, z_1)$ gives $rz = r_1z_1$, so (since $|z| = |z_1| = 1$) $r = |rz| = |r_1z_1| = r_1$. Then $z = z_1$, so $(r, z) = (r_1, z_1)$. Next σ is onto because $z = re^{i\theta}$ in polar form, so (since $\left|e^{i\theta}\right| = 1$) $z = \sigma(r, e^{i\theta})$. Finally, σ is an isomorphism because

   $$\sigma(r, z)\sigma(r_1, z_1) = (rz)(r_1z_1) = rr_1zz_1 = \sigma(rr_1, zz_1) = \sigma[(r, z), (r_1, z_1)].$$

4. Page 137, #15. *If H and K are subgroups of a group and $|H|$ is prime. Show that either $H \subseteq K$ or $H \cap K = \{1\}$.*

   Solution. Since $H \cap K \subseteq H$, we have that $|H \cap K|$ divides the prime $|H|$. So either $|H \cap K| = 1$ (and $H \cap K = \{1\}$) or $|H \cap K| = |H|$ (so $H \cap K = H$, that is $H \subseteq K$).

5. Page 137, #17. *Let $|G| = p^2$ where p is a prime. Show that every proper subgroup of G is cyclic.*

   Solution. Let H be a subgroup of G, $H \neq G$. Then $|H| = 1$ or p by Lagrange's Theorem. If $|H| = 1$, then $H = \langle 1\rangle$. If $|H| = p$, then H is cyclic by Corollary 3 of Lagrange's Theorem.

6. Page 153, #19. *If $H \lhd G$ and $K \lhd G$ define $HK = \{hk \mid h \in H, k \in K\}$. Show that HK is a subgroup of G and that $HK \lhd G$.*

   Solution. Clearly $1 = 1 \cdot 1 \in HK$. If hk and $h_1k_1 \in HK$ then $hk \cdot h_1k_1 = (hh_1)[(h_1^{-1}kh_1)k_1] \in HK$ because $K \lhd G$. Also $(hk)^{-1} = k^{-1}h^{-1} = h^{-1}(hk^{-1}h^{-1}) \in HK$, because $K \lhd G$. So HK is a subgroup (requiring only $K \lhd G$). Next $g(hk)g^{-1} = (ghg^{-1})(gkg^{-1}) \in HK$ for all $g \in G$, so $HK \lhd G$.