PMAT 315 Winter 2008

MIDTERM SOLUTIONS

[10] 1. Define the operation $*$ on the set $\mathbb{Z}$ of all integers by: for all $a, b \in \mathbb{Z}$, $a*b = a+b+1$.

(a) Prove that $(\mathbb{Z}, *)$ is a group.

Closure: For all $a, b \in \mathbb{Z}$, $a * b = a + b + 1 \in \mathbb{Z}$.
Associativity: For all $a, b, c \in \mathbb{Z}$,

$$(a*b)*c = (a+b+1)*c = a+b+1+c+1 = a+(b+c+1)+1 = a*(b+c+1) = a*(b*c).$$

Identity: $-1 \in \mathbb{Z}$ is the identity element, because

$$a * (-1) = a + (-1) + 1 = a \quad \text{and} \quad (-1) * a = (-1) + a + 1 = a \quad \text{for all } a \in \mathbb{Z}.$$

Inverses: For any $a \in \mathbb{Z}$, the inverse of a is $-a - 2 \in \mathbb{Z}$, because

$$a * (-a - 2) = a + (-a - 2) + 1 = -1 \quad \text{and} \quad (-a - 2) * a = (-a - 2) + a + 1 = -1,$$

and -1 is the identity.

Therefore $(\mathbb{Z}, *)$ is a group.

(b) Define the function $\phi : \mathbb{Z} \to \mathbb{Z}$ by: $\phi(a) = a - 1$ for all $a \in \mathbb{Z}$. Prove that ϕ is an isomorphism between the groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}, *)$.

ϕ is one-to-one: Let $a, b \in \mathbb{Z}$ so that $\phi(a) = \phi(b)$. This says that $a - 1 = b - 1$, so $a = b$. Thus ϕ is one-to-one.
ϕ is onto: Let $b \in \mathbb{Z}$ be arbitrary. Then $\phi(b + 1) = (b + 1) - 1 = b$ where $b + 1 \in \mathbb{Z}$, so ϕ is onto.
Finally, since $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, *)$, we must prove that $\phi(a + b) = \phi(a) * \phi(b)$ for all $a, b \in \mathbb{Z}$. Let $a, b \in \mathbb{Z}$ be arbitrary. Then

$$\phi(a) * \phi(b) = (a - 1) * (b - 1) = (a - 1) + (b - 1) + 1 = (a + b) - 1 = \phi(a + b).$$

Therefore $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, *)$ is an isomorphism.
Note. The group $(\mathbb{Z}, *)$ is simply what $(\mathbb{Z}, +)$ turns into if you relabel every integer a by $a - 1$ (and relabel $+$ by $*$). Thus for example $4 + 5 = 9$ becomes $3 * 4 = 8$.

[15] 2. Let $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 6 & 4 & 8 & 9 & 2 & 3 & 1 & 5 \end{bmatrix}$.

(a) Write σ as a product of disjoint cycles. Is σ even or odd? Explain.

$\sigma = (17348)(26)(59)$. Since (17348) (having length 5) is the product of four transpositions (for instance $(18)(14)(13)(17)$), σ is the product of six transpositions and therefore is an **even** permutation.

(b) Find the inverse σ^{-1} of σ in the symmetric group S_9, and write σ^{-1} as a product of disjoint cycles.

The inverse of a cycle is just that cycle written in reverse order. Thus

$$\sigma^{-1} = \Big((17348)(26)(59)\Big)^{-1} = (59)^{-1}(26)^{-1}(17348)^{-1} = (95)(62)(84371).$$

Since the cycles in σ^{-1} are disjoint, they commute, and σ^{-1} could be written in other ways such as (18437)(26)(59).

(c) Find a permutation $\tau \in S_9$ so that $\sigma\tau = (123)$.

$\sigma\tau = (123)$ is equivalent to $\tau = \sigma^{-1}\sigma\tau = \sigma^{-1}(123)$. Thus from part (b),

$$\tau = \sigma^{-1}(123) = (18437)(26)(59)(123) = (1627)(384)(59).$$

(d) Find the order of the subgroup $<\sigma>$ (of S_9) generated by σ. Explain.

The order of a product of disjoint cycles is the lcm of the orders (lengths) of the cycles. Thus the order of $\sigma = (17348)(26)(59)$ is $\text{lcm}(5, 2, 2) = 10$.

(e) Find two groups H_1 and H_2, both with order > 1, so that $<\sigma> \approx H_1 \times H_2$. Explain.

Since the order of σ is 10, $<\sigma> \approx \mathbb{Z}_{10}$. Since $10 = 5 \times 2$ and $\gcd(5, 2) = 1$, we know from class (or Corollary 2 on page 157) that $\mathbb{Z}_{10} \approx \mathbb{Z}_5 \oplus \mathbb{Z}_2$. Thus $<\sigma> \approx \mathbb{Z}_5 \oplus \mathbb{Z}_2$ and we could take $H_1 = \mathbb{Z}_5$ and $H_2 = \mathbb{Z}_2$.

[10] 3. Suppose that G is a group, g is an element of G, and H is a subgroup of G. Define the set $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.

(a) Prove that gHg^{-1} is a subgroup of G.

gHg^{-1} is obviously nonempty. Using the one-step subgroup test, we let $a_1 = gh_1g^{-1}$ and $a_2 = gh_2g^{-1}$ be arbitrary elements of gHg^{-1}, where $h_1, h_2 \in H$. Then

$$\begin{aligned} a_1a_2^{-1} &= (gh_1g^{-1})(gh_2g^{-1})^{-1} \\ &= gh_1g^{-1}(g^{-1})^{-1}h_2^{-1}g^{-1} = gh_1g^{-1}gh_2^{-1}g^{-1} = g(h_1h_2^{-1})g^{-1} \quad \in gHg^{-1}, \end{aligned}$$

where $h_1h_2^{-1} \in H$ since H is a subgroup.

(b) Suppose G is Abelian. Prove that $gHg^{-1} = H$.

Since G is Abelian, $ghg^{-1} = gg^{-1}h = h$ for all $g \in G$ and $h \in H$. Thus

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\} = \{h \mid h \in H\} = H.$$

(c) Find an example of a group G, an element $g \in G$, and a subgroup H of G so that $gHg^{-1} \neq H$.

We need G to be nonabelian, so try $G = S_3$. We need H to be proper and nontrivial, so try $H = \{\varepsilon, (12)\}$. Then with $g = (123)$ we get $g^{-1} = (132)$ and so

$$gHg^{-1} = \{(123)\varepsilon(132), (123)(12)(132)\} = \{\varepsilon, (23)\} \neq H.$$

$\overline{[35]}$