PMAT 315 L01 Winter 2007

FINAL EXAMINATION

[9] 1. (a) State Cayley's Theorem for groups.

(b) Let G be a group of order n. Show how to define an isomorphism ϕ from G to a subgroup of the symmetric group S_n. (You do not need to prove that your ϕ is an isomorphism.)

(c) Find a subgroup of S_5 which is isomorphic to the group $\mathbb{Z}_5$.

[16] 2. Let $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 8 & 6 & 9 & 5 \end{bmatrix} \in S_9$.

(a) Write σ as a product of disjoint cycles. Is σ an element of the alternating group A_9? Explain.

(b) Find the order of σ.

(c) Find σ^2 and σ^{-1}, and write them as products of disjoint cycles.

(d) Find $\tau \in S_9$ so that $\sigma\tau = (45)$. Write τ as a product of disjoint cycles.

[7] 3. Prove that as groups, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_6$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \not\approx \mathbb{Z}_8$.

[17] 4. Let $\mathbb{Z}[x]$ be the set of all polynomials with variable x and with integer coefficients, and let N be the set of all polynomials in $\mathbb{Z}[x]$ with zero constant term.

(a) $\mathbb{Z}[x]$ with addition is an Abelian group. Prove that N is a subgroup of $\mathbb{Z}[x]$.

(b) By part (a) the factor group $\mathbb{Z}[x]/N$ is defined. Describe the elements of this factor group. Give an isomorphism between $\mathbb{Z}[x]/N$ and $(\mathbb{Z}, +)$.

(c) Show that the function $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $\phi(f(x)) = 2f(0)$ for all $f(x) \in \mathbb{Z}[x]$ is a (group) homomorphism. Find the kernel of ϕ and the image of ϕ.

(d) Show that, considering $\mathbb{Z}[x]$ and $\mathbb{Z}$ as **rings** (with addition and multiplication), the function $\phi : \mathbb{Z}[x] \to \mathbb{Z}$ in part (c) is *not* a (ring) homomorphism.

[7] 5. (a) Let G be a cyclic group and let H be a (normal) subgroup of G. Prove that G/H is cyclic.

(b) Suppose that G is a group and H is a normal subgroup of G so that both H and G/H are cyclic. Is it necessarily true that G must be cyclic? Explain.

[8] 6. Let $M_2(\mathbb{Z})$ be the ring of all 2 by 2 matrices with integer entries, and let $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{Z} \right\}$.

(a) Prove that D is a subring of $M_2(\mathbb{Z})$.

(b) Suppose that I is an ideal of $M_2(\mathbb{Z})$ so that $D \subseteq I$. Prove that $I = M_2(\mathbb{Z})$.

[11] 7. Let G be a group, and let $H = \{g^2 | g \in G\}$.

(a) Prove that H is a subgroup of G if G is Abelian.

(b) Give an example to show that H need not be a subgroup of G if G is not Abelian. Give another example to show that H can be a subgroup of G even if G is not Abelian.

(c) Prove that if H is a subgroup of G, then H is a normal subgroup of G.

[75]