PMAT 315
ASSIGNMENT 5 WINTER 2010

1. #14, §3.1. *Given r and s in a ring R, show that $1+rs$ is a unit in R if and only if $1+sr$ is a unit.*
   [Hint: Compute $(1+sr)s(1+rs)^{-1}r$.] 8 marks

2. *A subset L of a ring R is called a* **left ideal** of *R if $RL \subseteq L$; that is if $rl \in L$ for all $r \in R$ and $l \in L$. If $R \neq 0$ and 0 and R are the only left ideals of R, show that R is a division ring.* 8 marks

3. #34 §3.3. *A ring R is called* **local** *if the set $J(R)$ of all non-units is an ideal.*
   (a) *Show that every division ring is local.* 1 mark
   (b) *If p is a prime show that $\mathbb{Z}_4$ is local ring that is not a field.* [*In fact, $\mathbb{Z}_{p^n}$ is local, p any prime.*] 1 mark
   (c) *If R is local, show that $R/J(R)$ is a division ring.* 1 mark
   (d) *If R is local, show that every ideal A of R, $A \neq R$, is contained in $J(R)$.* 1 mark
   (e) *If R is local and $A \subseteq J(R)$ is an ideal, show that R/A is local and $J(R/A) = \{r + A \mid r \in J(R)\}$.* 4 marks

4. #36 §3.4. *Let R be a ring and let η be a symbol. As in the discussion preceding Example 5 §3.2, let $R(\eta)$ denote the set of formal sums $a + b\eta$, $a, b \in R$.* Decree: *$\eta^2 = 0$; $a\eta = \eta a$ for all $a \in R$; and $a + b\eta = a' + b'\eta$ implies $a = a'$ and b'.*
   (a) *If A is an ideal of R, write $A(\eta) = \{a + b\eta \mid a, b \in A\}$. Show that $A(\eta)$ is an ideal of $R(\eta)$ and that*
   $R(\eta)/A(\eta) \cong (R/A)(\eta)$. 4 marks
   (b) *Show that* $R(\eta) \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in R \right\}$. 4 marks

5. #11 §3.4. *Show that $x^3 - 8x^2 + 5x + 3$ has no solution $x \in \mathbb{Z}$.* 8 marks

Total 40 marks