PMAT 315
SOLUTIONS TO ASSIGNMENT 3
WINTER 2010

1. §2.5, #28. *Show that* $\mathbb{R}^+ \times \mathbb{C}^0 \cong \mathbb{C}^*$ *where* $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$ *and* $\mathbb{C}^0 = \{z \in \mathbb{C} \mid |z| = 1\}$. [Hint: If $z \in C^*$ then $z = re^{i\theta}$ where $r > 0$.] 8 marks

   SOLUTION. Define $\sigma : \mathbb{R}^+ \times \mathbb{C}^0 \to \mathbb{C}^*$ by $\sigma(r, z) = rz$.

   (a) σ is one-to-one. If $\sigma(r, z) = \sigma(r_1, z_1)$, then $rz = r_1 z_1$. Since $|z| = |z_1| = 1$, this gives $r = |rz| = |r_1 z_1| = r_1$. But then $z = z_1$, so $(r, z) = (r_1, z_1)$, proving that σ is one-to-one.

   (b) σ is onto. Given $z \in \mathbb{C}^*$, write $z = re^{i\theta}$ in polar form where $r > 0$. We have $e^{i\theta} \in \mathbb{C}^0$ because $\left|e^{i\theta}\right| = 1$, so $z = \sigma(r, e^{i\theta})$. This proves that σ is onto.

   (c) σ is a homomorphism. Given (r, z) and (r_1, z_1) in $\mathbb{R}^+ \times \mathbb{C}^0$ we compute:
$$\sigma(r, z)\sigma(r_1, z_1) = (rz)(r_1 z_1) = rr_1 zz_1 = \sigma(rr_1, zz_1) = \sigma[(r, z)(r_1, z_1)].$$
   Hence σ is a homomorphism, and so it is an isomorphism by (a) and (b).

2. *Show that* $(\mathbb{R}, +)$ *is not isomorphic to* $\mathbb{R}^*$. [Hint: If $\sigma : \mathbb{R} \to \mathbb{R}^*$ is an isomorphism, there exists $r \in \mathbb{R}$ such that $\sigma(r) = -1$.] 8 marks

   SOLUTION. If $\sigma(r) = -1$ as in the Hint, put $a = \sigma(\frac{1}{2}r)$. Then, because σ is a homomorphism,
$$-1 = \sigma(r) = \sigma(\tfrac{1}{2}r + \tfrac{1}{2}r) = \sigma(\tfrac{1}{2}r) \cdot \sigma(\tfrac{1}{2}r) = a^2.$$
   This is a contradiction as a is a real number, so no such isomorphism σ exists.

   Remark: The solution requires only that σ is an onto homomorphism.

3. *Let H and K be finite subgroups of some group G.*

   (1) *If $|H|$ and $|K|$ are relatively prime, show that $H \cap K = \{1\}$.* 6 marks

   (2) *If $H \cap K = \{1\}$, does it follow that $|H|$ and $|K|$ are relatively prime? Justify your answer.* 2 marks

   SOLUTION. Write $|H| = m$ and $|K| = n$.

   (1). Observe that $H \cap K$ is a subgroup of H. Since $H \cap K \subseteq H$ it follows by Lagrange's theorem that $|H \cap K|$ divides m. Similarly $|H \cap K|$ divides n. Since $gcd(m, n) = 1$ by hypothesis, it follows that $|H \cap K| = 1$; that is $H \cap K = \{1\}$.

   (2). Take $H = \{\varepsilon, (1\ 2)\}$ and $K = \{\varepsilon, (1\ 3)\}$, subgroups of S_3. Then $H \cap K = \{\varepsilon\}$, but $|H| = 2 = |K|$.

4. §2.4, #26(a). *Let $|g| = m$ and $|h| = n$ in a group G where m and n are relatively prime. If $gh = hg$, show that $|gh| = mn$. Is $|gh| = lcm(m, n)$ in general?* 8 marks

   SOLUTION. Write $|gh| = d$. Since $gh = hg$, we have $(gh)^{mn} = g^{mn}h^{mn} = (g^m)^n(h^n)^m = 1$. This means $d \mid mn$. To prove $mn \mid d$, it suffices to show $m \mid d$ and $n \mid d$ (by Theorem 5 §1.2 because $\gcd(m, n) = 1$). This in turn follows if we can show $g^d = 1$ and $h^d = 1$. We have $1 = (gh)^d = g^d h^d$, so $g^d = h^{-d} \in \langle g\rangle \cap \langle h\rangle$. But $\langle g\rangle \cap \langle h\rangle = \{1\}$ by Lagrange's theorem because $|\langle g\rangle| = m$, $|\langle h\rangle| = n$ and $\gcd(m, n) = 1$ (see the preceding exercise). Thus $g^d = 1$ and $h^{-d} = 1$, so $h^d = 1$. This is what we wanted.

   If $\gcd(m, n) \neq 1$, nothing can be said (for example, consider the case $h = g^{-1}$).

5. *If a group G has subgroups or orders 45 and 75, and if $|G| \leq 400$, determine $|G|$.* 8 marks

   SOLUTION. Let H and K be subgroups of G with $|H| = 45$ and $|K| = 75$. Write $|G| = n$, so $45 \mid n$ and $75 \mid n$ by Lagrange's theorem. Consequently $lcm(45, 75)$ divides n, that is $225 \mid n$. Thus n is one of 225, 450, 675, $\cdots$. Since $n = |G| \leq 400$ by hypothesis, it follows that $|G| = n = 225$.

Total 40 marks