PMAT 315
SOLUTIONS TO ASSIGNMENT 4
WINTER 2010

1. *Let G be a group with $|G| = p^2$ where p is a prime. Show that G is abelian. You may use the known fact that $Z(G) \neq \{1\}$ for such a group.* 8 marks

   SOLUTION. Write $Z(G) = Z$ for convenience. We must show that $Z = G$. We know that $Z \neq \{1\}$, so there are two cases:

   (a) $|Z| = p^2$. Then $Z = G$ so G is abelian.

   (b) $|Z| = p$. Then $|G/Z| = \frac{|G|}{|Z|} = \frac{p^2}{p} = p$, so G/Z is cyclic. Hence G is abelian by Theorem 2 §2.9.

2. *Write $G = D_6 = \{1, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$ where, as usual $|a| = 6$, $|b| = 2$ and $aba = b$. Consider the subgroup $K = \langle a^2 \rangle = \{1, a^2, a^4\}$.*

   (1) *Show that $K \lhd G$.* 4 marks

   (2) *Show that $G' = K$.* 4 marks

   SOLUTION. (1). We give two proofs that $K \lhd G$. [Remark: $K \neq Z(G)$ as for D_4 as $Z(G) = \{1, a^3\}$.]

   (a). K is a characteristic subgroup of the cyclic subgroup $\langle a \rangle$, and $\langle a \rangle \lhd D_4$, so $K \lhd G$ by results in class.

   (b). Clearly $a^t K a^{-t} = K$ for each t. We have $(ba^t)K(ba^t)^{-1} \subseteq K$ because $(ba^t)a^2(ba^t)^{-1} = a^4$ and $(ba^t)a^4(ba^t)^{-1} = a^2$.

   (2). We have $\left|\frac{G}{K}\right| = \frac{|G|}{|K|} = \frac{12}{3} = 4$, so $\frac{G}{K}$ is an abelian group. Hence $G' \subseteq K$ by Theorem 3 §2.9. But $|K| = 3$ is a prime, so either $G' = \{1\}$ or $G' = K$. But $G' \neq \{1\}$ because G is not abelian.

3. §2.9, #7. *Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite abelian group in which every element has finite order.* 8 marks

   SOLUTION. Let $q = \frac{m}{n}$ be any element of $\mathbb{Q}$. Then $nq = m \in \mathbb{Z}$, so $|q|$ divides n. In particular, $|q|$ is finite.

4. §2.10, #17. *Let G be an abelian group. Let $T(G)$ denote the set of elements of G of finite order, and call G* **torsion-free** *if $T(G) = \{1\}$, that is if 1 is the only element of finite order.*

   (1) *Show that $T(G)$ is a subgroup of G.* 4 marks

   (2) *Show that $G/T(G)$ is torsion-free for any abelian group G.* 4 marks

   SOLUTION. (1). If $|g| = m$ and $|h| = n$ then $\left|g^{-1}\right| = m$. As G is abelian, $(gh)^{mn} = g^{mn}h^{mn} = (g^m)^n(h^n)^m = 1$. It follows that $|gh|$ is finite.

   (2) Write $T = T(G)$ for convenience. If $Ta \in G/T$ has finite order $n > 1$, we must show that $Ta = T$. We have $T = (Ta)^n = Ta^n$ so $a^n \in T$. But every element of T has finite order, so $(a^n)^m = 1$ for some $m > 1$. Hence $a^{mn} = 1$ where $mn > 1$ so a has finite order. Hence $a \in T$, that is $Ta = T$, as required.

5. In each case use the isomorphism theorem.

   (1) Let $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R}, a \neq 0 \neq c \right\}$, *a group using matrix multiplication. If $K = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$, show that $K \lhd G$ and $G/K \cong \mathbb{R}^* \times \mathbb{R}^*$.* 4 marks

   (2) §2.10, # 21. *Show that $\mathbb{C}^*/\mathbb{C}^0 \cong \mathbb{R}^+$ where $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$ under multiplication.* 4 marks

   SOLUTION. (1). G is a subgroup of the group $GL_2(\mathbb{R})$ of all invertible 2×2 matrices over $\mathbb{R}$. Define a map $\alpha : G \to \mathbb{R}^* \times \mathbb{R}^*$ by $\alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = (a, c)$. Then α is a group homomorphism because

   $$\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \right) = \alpha \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix} = (aa', cc') = (a, c)(a', c') = \alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \alpha \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}.$$

   We have $ker(\alpha) = K$ because $(1, 1)$ is the unity of $\mathbb{R}^* \times \mathbb{R}^*$. Since $im(\alpha) = \mathbb{R}^* \times \mathbb{R}^*$, the isomorphism theorem shows that $K \lhd G$ and $G/K \cong im(\alpha) = \mathbb{R}^* \times \mathbb{R}^*$.

Total 40 marks