PMAT 315
SOLUTIONS TO ASSIGNMENT 5
WINTER 2010

1. #14, §3.1. *Given r and s in a ring R, show that $1+rs$ is a unit in R if and only if $1+sr$ is a unit.*

   [Hint: Compute $(1+sr)[1-s(1+rs)^{-1}r$.] 8 marks

   Solution. As in the Hint, we have $(1+sr)s(1+rs)^{-1}r = s(1+rs)(1+rs)^{-1}r = sr$. So $(1+sr)[1-s(1+rs)^{-1}r] = 1$. Similarly $[1-s(1+rs)^{-1}r](1+sr) = 1$.

2. *A subset L of a ring R is called a* **left ideal** *of R if $RL \subseteq L$; that is if $rl \in L$ for all $r \in R$ and $l \in L$. If $R \neq 0$ and 0 and R are the only left ideals of R, show that R is a division ring.* 8 marks

   Solution. If $0 \neq a \in R$, we must show that a is invertible. Consider $Ra = \{ra \mid a \in R\}$. This is a left ideal and $Ra \neq 0$ (because $a \in Ra$). Hence $Ra = R$ by hypothesis, so $ba = 1$ for some $b \in R$. Since $b \neq 0$ (because $R \neq 0$), the same argument gives $Rb = R$, and so $cb = 1$ for some $c \in R$. But then $c = c1 = c(ba) = (cb)a = 1a = a$. Hence $cb = 1$ becomes $ab = 1$, so $b = a^{-1}$.

3. #34 §3.3. *A ring R is called* **local** *if the set $J(R)$ of all non-units is an ideal.*

   (a) *Show that every division ring is local.* 1 mark

   (b) *Show that $\mathbb{Z}_4$ is local ring that is not a field. [In fact, $\mathbb{Z}_{p^n}$ is local, p any prime.]* 1 mark

   (c) *If R is local, show that $R/J(R)$ is a division ring.* 1 mark

   (d) *If R is local, show that every ideal A of R, $A \neq R$, is contained in $J(R)$.* 1 mark

   (e) *If R is local and $A \subseteq J(R)$ is an ideal, show that R/A is local and $J(R/A) = \{r + A \mid r \in J(R)\}$.* 4 marks

   Solution. (a). Here $J(R) = \{0\}$ is an ideal.

   (b). Write $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. Then $J(\mathbb{Z}_4) = \{\bar{0}, \bar{2}\}$ since $\bar{1}$ and $\bar{3}$ are units.

   (c). Write $J = J(R)$. If $a+J \neq 0$ in R/J then $a \notin J$ so a is a unit in R. But if $ab = 1$ then $(a+J)(b+J) = 1+J$, so $a+J$ is a unit in R/J.

   (d). If $A \not\subseteq J(R)$ then A contains a unit u, so $r = (ru^{-1})u \in A$ for all $r \in R$. Hence $R = A$, contrary to choice.

   (e). Write $J = J(R)$. We show that $J(R/J)$ and $J/A = \{r + A \mid r \in J\}$. If $r + A$ is a nonunit in R/A then r is a nonunit in R ($rs = 1 \Rightarrow (r+A)(s+A) = 1 + A$). Hence $J[R/A] \subseteq J/A$. Conversely, let $r + A \in J/A$, so $r \in J$. We must show $r + A$ is a nonunit in R/A. Suppose not, and write $(r+A)^{-1} = s + A$. Then $rs - 1 \in A$ so $rs - 1 \in J$. But $r \in J$ and J is an ideal, so $rs \in J$. Thus $-1 \in J$, $1 \in J$, a contradiction.

4. #36 §3.4. *Let R be a ring and let η be a symbol. As in the discussion preceding Example 5 §3.2, let $R(\eta)$ denote the set of formal sums $a + b\eta$, $a, b \in R$. Decree: $\eta^2 = 0$; $a\eta = \eta a$ for all $a \in R$; and $a + b\eta = a' + b'\eta$ implies $a = a'$ and b'.*

   (a) *If A is an ideal of R, write $A(\eta) = \{a + b\eta \mid a, b \in A\}$. Show that $A(\eta)$ is an ideal of $R(\eta)$ and that*

   $R(\eta)/A(\eta) \cong (R/A)(\eta)$. 4 marks

   (b) *Show that* $R(\eta) \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in R \right\}$. 4 marks

   Solution. (a). Define $\theta : R(\eta) \to \frac{R}{A}(\eta)$ by $\theta(r + s\eta) = \bar{r} + \bar{s}\eta$ where $\bar{r} = r + A$. This is an onto ring homomorphism and $\ker\theta = \{r + s\eta | \bar{r} = \bar{s} = 1\} = A(\eta)$.

   (b). Define $\sigma : R(\eta) \to \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in R \right\}$ by $\sigma(a + b\eta) = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. This is an additive group isomorphism and $\sigma(a + b\eta) \cdot \sigma(c + d\eta) = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix} = \sigma(ac + (ad + bc)\eta) = \sigma[(a + b\eta)(c + d\eta)]$. So θ is a ring isomorphism.

5. #11 §3.4. *Show that $x^3 - 8x^2 + 5x + 3$ has no solution $x \in \mathbb{Z}$.* 8 marks

   Solution. In $\mathbb{Z}_4$ this becomes $x^3 + x - 1 = 0$. If $x = 0, 1, 2, -1$ then $x^3 + x - 1 = -1, 1, 1, 1$, respectively, so there is no solution in $\mathbb{Z}_4$. Hence there is no solution in $\mathbb{Z}$.