# PURE MATHEMATICS 319 WINTER 2004
## Assignment 1 Solutions

Let $l, m, n$ be distinct lines and $P, Q, R$ be distinct points.

(a) We prove that $\sigma_l \sigma_m = \sigma_m \sigma_l$ if and only if $l \perp m$.

*Proof*:

First, we prove that if $\sigma_l \sigma_m = \sigma_m \sigma_l$ then $l \perp m$. Suppose that $\sigma_l \sigma_m = \sigma_m \sigma_l$. Since $l \neq m$, there is a point $A \in l$ so that $A \notin m$. Then since $A \in l$, we have $\sigma_l(A) = A$ and so $\sigma_l(\sigma m(A)) = \sigma_l \sigma m(A) = \sigma_m \sigma_l(A) = \sigma_m(A)$. Thus, $\sigma_l$ fixes $\sigma m(A)$, and since $\sigma_l$ fixes only the points of $l$, we have $\sigma m(A) \in l$. Since $A \notin m$, $\sigma m(A) \neq A$ and from the definition of reflections, $m$ is the perpendicular bisector of $\overleftrightarrow{A\sigma m(A)}$ However, since $\sigma m(A)$ and $A$ are two dictinct points of $l$, $l = \overleftrightarrow{A\sigma m(A)}$ and so $l \perp m$.

Next, we prove that if $l \perp m$ then $\sigma_l \sigma_m = \sigma_m \sigma_l$. Suppose that $l \perp m$. Put $C = l \cap m$. From $l \perp m$ we get that $\sigma_l \sigma_m = \sigma_C = \sigma_C^{-1} = (\sigma_l \sigma_m)^{-1} = \sigma_m^{-1} \sigma_l^{-1} = \sigma_m \sigma_l$.

(b) We prove that $\sigma_P \sigma_m = \sigma_m \sigma_P$ if and only if $P \in m$.

*Proof*:

First, we prove that if $\sigma_P \sigma_m = \sigma_m \sigma_P$ then $P \in m$. Suppose that $\sigma_P \sigma_m = \sigma_m \sigma_P$. Then $\sigma_P(\sigma_m(P)) = \sigma_P \sigma_m(P) = \sigma_m \sigma_P(P) = \sigma_m(P)$. Thus, $\sigma_P$ fixes $\sigma_m(P)$ which implies that $\sigma_m(P) = P$ (because $\sigma_P$ fixes $P$ only). Since $\sigma_m(P) = P$ and $\sigma_m$ fixes only the points of $m$, we get that $P \in m$.

Next, we prove that if $P \in m$ then $\sigma_P \sigma_m = \sigma_m \sigma_P$. Suppose that $P \in m$. Let $l$ be a line through $P$ and perpendicular to $m$. Then $\sigma_P = \sigma_l \sigma_m = \sigma_m \sigma_l$ and so $\sigma_P \sigma_m = (\sigma_m \sigma_l) \sigma_m = \sigma_m(\sigma_l \sigma_m) = \sigma_m \sigma_P$.

(c) $\sigma_l \sigma_m \sigma_n = \sigma_n \sigma_m \sigma_l$ if and only if $l, m, n$ are either concurrent or parallel.

*Proof*:

First, we prove that if $\sigma_l \sigma_m \sigma_n = \sigma_n \sigma_m \sigma_l$ then $l, m, n$ are either concurrent or parallel. Suppose that $\sigma_l \sigma_m \sigma_n = \sigma_n \sigma_m \sigma_l$. Now, suppose that $l, m, n$ are not mutually parallel; that is, at least two of $l, m, n$ intersect. We show that $l, m, n$ must be concurrent.

**Case 1**: $l$ and $m$ intersect, that is, $l \cap m = A$.

Then $\sigma_l \sigma_m$ is a rotation centered at $A$ and from the fact that $A \in l$ and $A \in m$, we have

$\sigma_l \sigma_m(\sigma_n(A)) = \sigma_l \sigma_m \sigma_n(A) = \sigma_n \sigma_m \sigma_l(A) = \sigma_n \sigma_m(\sigma_l(A)) = \sigma_n \sigma_m(A) = \sigma_n(A)$.

This means that the rotation $\sigma_l \sigma_m$ fixes $\sigma_n(A)$, but a rotation only fixes its centre, so $\sigma_n(A) = A$. Thus, $\sigma_n$ fixes $A$, but $\sigma_n$ only fixes points on $n$, so $A \in n$. Hence, $A = l \cap m \cap n$, that is, $l, m, n$ are concurrent.

**Case 2**: $m$ and $n$ intersect. This is the same as Case 1.

**Case 3**: $l$ and $n$ intersect, that is, $l \cap m = A$.

Then from $A \in l$ and $A \in n$, we have $\sigma_l(A) = \sigma_n(A) = A$ and so

$$
\begin{aligned}
\sigma_n\sigma_l\left(\sigma_m\left(A\right)\right) &= \sigma_n\sigma_l\sigma_m\sigma_n\left(A\right) && \text{because } A=\sigma_n\left(A\right)\\
&= \sigma_n\sigma_n\sigma_m\sigma_l\left(A\right) && \text{because } \sigma_l\sigma_m\sigma_n=\sigma_n\sigma_m\sigma_l\\
&= \sigma_m\sigma_l\left(A\right) && \text{because } \sigma_n\sigma_n=i\\
&= \sigma_m\left(A\right) && \text{because } \sigma_l\left(A\right)=A.
\end{aligned}
$$

This means that the rotation $\sigma_n\sigma_l$ fixes $\sigma_m\left(A\right)$, but a rotation only fixes its centre, so $\sigma_m\left(A\right)=A$. Thus, $\sigma_m$ fixes $A$, but $\sigma_m$ only fixes points on $m$, so $A\in m$. Hence, $A=l\cap m\cap n$, that is, $l,m,n$ are concurrent.

Next, we prove that if $l,m,n$ are either concurrent or parallel then $\sigma_l\sigma_m\sigma_n=\sigma_n\sigma_m\sigma_l$. Suppose that $l,m,n$ are either concurrent or parallel. Then, as proven in the lectures, $\sigma_l\sigma_m\sigma_n=\sigma_t$ for some line $t$ and so $\sigma_l\sigma_m\sigma_n=\sigma_t=\sigma_t^{-1}=\left(\sigma_l\sigma_m\sigma_n\right)^{-1}=\sigma_n^{-1}\sigma_m^{-1}\sigma_l^{-1}=\sigma_n\sigma_m\sigma_l$.

(d) $\sigma_P\sigma_Q=\sigma_Q\sigma_R$ if and only if $Q$ is the midpoint of $\overline{PR}$.

*Proof*:

First, we prove that if $\sigma_P\sigma_Q=\sigma_Q\sigma_R$ then $Q$ is the midpoint of $\overline{PR}$. Suppose that $\sigma_P\sigma_Q=\sigma_Q\sigma_R$. Then $\sigma_P\left(\sigma_Q\left(R\right)\right)=\sigma_P\sigma_Q\left(R\right)=\sigma_Q\sigma_R\left(R\right)=\sigma_Q\left(R\right)$, which means that $\sigma_P$ fixes $\sigma_Q\left(R\right)$, and so $\sigma_Q\left(R\right)=P$ because $\sigma_P$ fixes $P$ only. Now, since $\sigma_Q\left(R\right)=P$ and $R\neq P$, by the definition of $\sigma_Q$, $Q$ is the midpoint of $\overline{PR}$.

Next, we prove that if $Q$ is the midpoint of $\overline{PR}$ then $\sigma_P\sigma_Q=\sigma_Q\sigma_R$. Suppose that $Q$ is the midpoint of $\overline{PR}$. Let $d$ be the line through $P$, $Q$ and $R$. Let $a,b,c$ be the lines perpendicular to $d$ at $P,Q,R$ respectively. Then, $\sigma_P=\sigma_a\sigma_d=\sigma_d\sigma_a$, $\sigma_Q=\sigma_b\sigma_d=\sigma_d\sigma_b$, and $\sigma_R=\sigma_c\sigma_d=\sigma_d\sigma_c$. Thus,

$$
\begin{aligned}
\sigma_P\sigma_Q &= \sigma_a\sigma_d\sigma_d\sigma_b && \text{because } \sigma_P=\sigma_a\sigma_d \text{ and } \sigma_Q=\sigma_d\sigma_b\\
&= \sigma_a\sigma_b && \text{because } \sigma_d\sigma_d=i\\
&= \tau_{PQ}^2 && \text{because } \sigma_a\sigma_b \text{ is the translation twice the distance from } b \text{ to } a\\
&= \tau_{QR}^2 && \text{because } Q \text{ is the midpoint of } \overline{PR}\\
&= \sigma_b\sigma_c && \text{because } \sigma_b\sigma_c \text{ is the translation twice the distance from } c \text{ to } b\\
&= \sigma_b\sigma_d\sigma_d\sigma_c && \text{because } \sigma_d\sigma_d=i\\
&= \sigma_Q\sigma_R && \text{because } \sigma_Q=\sigma_b\sigma_d \text{ and } \sigma_R=\sigma_d\sigma_c
\end{aligned}
$$

(e) $\sigma_P\sigma_l=\sigma_l\sigma_Q$ if and only if $l$ is the perpendicular bisector of $\overline{PQ}$.

*Proof*:

First, we prove that if $\sigma_P\sigma_l=\sigma_l\sigma_Q$ then $l$ is the perpendicular bisector of $\overline{PQ}$. Suppose that $\sigma_P\sigma_l=\sigma_l\sigma_Q$. Then $\sigma_P\left(\sigma_l\left(Q\right)\right)=\sigma_l\sigma_Q\left(Q\right)=\sigma_l\left(Q\right)$. Thus, $\sigma_P$ fixes $\sigma_l\left(Q\right)$, and so $\sigma_l\left(Q\right)=P$ because $\sigma_P$ fixes $P$ only. Since $\sigma_l\left(Q\right)=P$ and $P\neq Q$, by the definition of $\sigma_l$, $l$ is the perpendicular bisector of $\overline{PQ}$.

Next, we prove that if $l$ is the perpendicular bisector of $\overline{PQ}$ then $\sigma_P\sigma_l=\sigma_l\sigma_Q$. Suppose that $l$ is the perpendicular bisector of $\overline{PQ}$. Let $c$ be the line through $P$ and $Q$. Let $a$ and $b$ be the lines perpendicular to $c$ at $P$ and $Q$ respectively. Put $A=l\cap c$. Since the directed distance from $l$ to $a$ equals the directed distance from $b$ to $l$, we have $\sigma_a\sigma_l=\sigma_l\sigma_b$. Also, $\sigma_P=\sigma_c\sigma_a$, $\sigma_Q=\sigma_c\sigma_b$ and $\sigma_l\sigma_c=\sigma_A=\sigma_c\sigma_l$. Thus,

$$\begin{aligned}
\sigma_P\sigma_l &= \sigma_c\sigma_a\sigma_l && \text{because } \sigma_P = \sigma_c\sigma_a \\
&= \sigma_c\sigma_l\sigma_b && \text{because } \sigma_a\sigma_l = \sigma_l\sigma_b \\
&= \sigma_l\sigma_c\sigma_b && \text{because } \sigma_l\sigma_c = \sigma_c\sigma_l \\
&= \sigma_l\sigma_Q && \text{because } \sigma_Q = \sigma_c\sigma_b
\end{aligned}$$