PURE MATHEMATICS 319 WINTER 2004
Assignment 2 Solutions

6.16 Show that the product of the reflections in the three angle bisectors of a triagnle is a reflection in a line perpendicular to a side of the triangle.

Proof: Let A, B, C be the vertices of a triangle and let a, b, c be the angle bisectors of the angles at A, B, C respectively. Then a, b, c are concurrent at a point D inside the triangle. Since a, b, c are concurrent at D, $\sigma_a\sigma_b\sigma_c = \sigma_d$ for some line d where $D \in d$. We prove that $d \perp \overleftrightarrow{AC}$. Now,

$$\begin{aligned}
\sigma_d\left(\overleftrightarrow{AC}\right) &= \sigma_a\sigma_b\sigma_c\left(\overleftrightarrow{AC}\right) \\
&= \sigma_a\sigma_b\left(\overleftrightarrow{BC}\right) && \text{since } c \text{ is the angle bisector of } \angle BCA \\
&= \sigma_a\left(\overleftrightarrow{AC}\right) && \text{since } b \text{ is the angle bisector of } \angle CBA \\
&= \overleftrightarrow{AC} && \text{since } a \text{ is the angle bisector of } \angle BAC.
\end{aligned}$$

Thus, σ_d fixes $\overleftrightarrow{AC}$, and so $d = \overleftrightarrow{AC}$ or $d \perp \overleftrightarrow{AC}$. However, $d \neq \overleftrightarrow{AC}$ because $D \in d$ and $D \notin \overleftrightarrow{AC}$, and therefore, $d \perp \overleftrightarrow{AC}$.

6.17 If $\sigma_m\sigma_n(x, y) = (x + 6, y - 3)$, find equations of the lines m and n.

Solution: Since $\sigma_m\sigma_n(x, y) = (x + 6, y - 3)$, $\sigma_m\sigma_n$ is a translation, say $\sigma_m\sigma_n = \tau$. Let $P = (0, 0)$. Then $\tau(P) = (6, -3)$, and we can choose m and n be the line perpendicular to $\overleftrightarrow{P\tau(P)}$ at M and P respectively, where M is the midpoint of $\overline{P\tau(P)}$, that is, $M = \frac{1}{2}(P + \tau(P)) = \left(3, -\frac{3}{2}\right)$. Now, the slope of $\overleftrightarrow{P\tau(P)}$ is $-\frac{3}{6} = -\frac{1}{2}$ and so the slope of m and n is 2. Thus, an equation of n is $y - 0 = 2(x - 0)$ or $y = 2x$, and an equation of m is $y - \left(-\frac{3}{2}\right) = 2(x - 3)$ or $y = 2x - \frac{15}{2}$.

6.18 If l and m are distinct intersecting lines, find the locus of all points P such that $\rho_{P,\theta}(l) = m$ for some θ.

Solution: Let l and m are distinct intersecting lines and $l \cap m = C$. We show that the locus of all points P such that $\rho_{P,\theta}(l) = m$ for some θ is the union of the two angle bisectors of the angles between l and m.

First, let P be a point on the angle bisector of an angle between l and m. We show that there is θ so that $\rho_{P,\theta}(l) = m$. Choose two lines a and b so that $P \in a$, $P \in b$, $a \perp l$ and $b \perp m$. Put $A = a \cap l$, $B = b \cap m$ and put θ to be the directed angle from l to m. Then it is easy to see that $\rho_{P,\theta}(l) = m$.

Next, suppose that $\rho_{P,\theta}(l) = m$. We show that P is a point on the angle bisector of an angle between l and m. Since $\rho_{P,\theta}$ is onto and $\rho_{P,\theta}(l) = m$, there is a point $A \in l$ so that $\rho_{P,\theta}(A) = C$. If $A = C$ then from $\rho_{P,\theta}(C) = C$ we have $P = C$ and so P is a point on the angle bisector of an angle between l and m. Now, suppose that $A \neq C$. Put B to be the midpoint of $\overline{AC}$, and put $D = \rho_{P,\theta}(B)$. Since $\rho_{P,\theta}$ is an isometry,

$CD = \rho_{P,\theta}(A)\,\rho_{P,\theta}(B) = AB = BC$. Now, since $D = \rho_{P,\theta}(B)$, P is on the perpendicular bisector of $\overline{BD}$, which is the angle bisector of $\angle BCD$ (because $CD = CB$) which is the angle bisector of an angle between l and m.

7.14 Show that if $\rho_1,\ \rho_2,\ \rho_2\rho_1$ and $\rho_2^{-1}\rho_1$ are rotations then the centres of $\rho_1,\ \rho_2\rho_1$ and $\rho_2^{-1}\rho_1$ are colinear.

Proof: Suppose that $\rho_1,\ \rho_2,\ \rho_2\rho_1$ and $\rho_2^{-1}\rho_1$ are rotations. Let A and B be the centres of ρ_1 and ρ_2 respectively. Let l be a line that passes through A and B. Let m be a line that passes through A so that $\rho_1 = \sigma_l\sigma_m$ and let n be a line that passes through B so that $\rho_2 = \sigma_n\sigma_l$. Now, since $\rho_2\rho_1 = \sigma_n\sigma_l\sigma_l\sigma_m = \sigma_n\sigma_m$ is a rotation, $n \cap m = C$ is the centre of the rotation $\rho_2\rho_1$. Similarly, $\rho_2^{-1}\rho_1 = (\sigma_n\sigma_l)^{-1}\sigma_l\sigma_m = \sigma_l\sigma_n\sigma_l\sigma_m = \sigma_{\sigma_l(n)}\sigma_m$ is a rotation, $\sigma_l(n) \cap m = D$ is the centre of the rotation of $\rho_2^{-1}\rho_1$. Then, the centres of $\rho_1,\ \rho_2\rho_1$ and $\rho_2^{-1}\rho_1$ are colinear because they are A, C and D, which are on the line m.

7.17 Show that $\sigma_P\sigma_l\sigma_P\sigma_l\sigma_P\sigma_l\sigma_P$ is a reflection in a line parallel to the line l.
Proof: Let P be a point and l be a line. Then $\sigma_P\sigma_l\sigma_P = \sigma_P\sigma_l(\sigma_P)^{-1} = \sigma_{\sigma_P(l)}$ and so

$$\begin{aligned}\sigma_P\sigma_l\sigma_P\sigma_l\sigma_P\sigma_l\sigma_P &= (\sigma_P\sigma_l\sigma_P)\,\sigma_l\,(\sigma_P\sigma_l\sigma_P)\\ &= \sigma_{\sigma_P(l)}\sigma_l\sigma_{\sigma_P(l)}\\ &= \sigma_{\sigma_P(l)}\sigma_l\sigma_{\sigma_P(l)}^{-1}\\ &= \sigma_{\sigma_{\sigma_P(l)}(l)}\end{aligned}$$

Thus, $\sigma_P\sigma_l\sigma_P\sigma_l\sigma_P\sigma_l\sigma_P$ is the reflection in the line $\sigma_{\sigma_P(l)}(l)$, so we only need to show that $\sigma_{\sigma_P(l)}(l)$ is parallel to l.

Put $n = \sigma_P(l)$. Since σ_P is a dilatation (which maps a line to a parallel line), we have $n \parallel l$. Now, put $m = \sigma_n(l)$. Since σ_n is a collineation (which maps parallel lines to parallel lines), we have $\sigma_n(n) \parallel \sigma_n(l)$; that is, $n \parallel m$. Since $n \parallel m$ and $n \parallel l$, we have $m \parallel l$. However, $m = \sigma_n(l) = \sigma_{\sigma_P(l)}(l)$. Thus, we have shown that $\sigma_{\sigma_P(l)}(l)$ is parallel to l.