Fall 2008 PMAT 415 L0 test 4 14:00. 18.11.2008

time: 30minutes

NO CALCULATORS

ID NUMBER:

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Name:	ID:

Problem 1: [5 pts] Find a set $\mathcal{A}$ of open intervals in $\mathbb{R}$ such that every rational number belongs to one of those intervals, but $\bigcup \mathcal{A} \neq \mathbb{R}$. (Hint: Make the length of those intervals shorter and shorter so that their sum is less than or equal to 1.)

Let $r_1, r_2, r_3, \ldots$ be an ω-enumeration of the rational numbers. For every $i \in \mathbb{N}$ let $\epsilon_i = \frac{1}{2}\frac{1}{2^i}$ and let I_i be the open interval $I_i = (r_i - \epsilon_i, r_i + \epsilon_i)$. The length of the interval I_i is $\frac{1}{2^i}$. Let $\mathcal{A} = \{I_i : i \in \mathbb{N}\}$.

Note that every rational number is in at least one of the intervals in $\mathcal{A}$ and that the intervals in $\mathcal{A}$ do not cover the real line because the sum of their lengths is $\sum_{i \in \mathbb{N}} \frac{1}{2^i} = 1$.

Other solution:

Let r be a real non rational number and $\mathcal{A} = \{\big(ne, (n+1)e\big) : \ n \text{ is an integer}\} \cup \{(0,1)\}$.

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Problem 2: [5 pts] Assume that $<$ is a well ordering on A and that $f : A \to A$ satisfies the condition

$$x < y \quad \text{implies} \quad f(x) < f(y)$$

for all x and y in A. Show that $x \leq f(x)$ for all $x \in A$. (Hint: Consider $f(f(x))$.

Assume for a contradiction that there is an element $x \in A$ with $f(x) < x$.
Then $f(f(x) < f(x)$. That is $f^{n+1}(x) < f^n(x)$ for all $n \in \omega$.
The sequence $x > f(x) > f^2(x) > f^3(x) > f^4(x) > \ldots$ is then an infinite decreasing sequence of elements of A in contradiction to the assumption that $(A; <)$ is a well order.