UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS & STATISTICS
PMAT 421

FINAL EXAMINATION

TIME: 2 HOURS

The marks for each problem are to the left of the problem number. Total marks [75].

[6] 1. Let $f : \mathbf{C} \to \mathbf{C}$ be defined by $f(z) = x^2 + y^2 + 2xyi$. Find all complex numbers $z = x + iy$ so that f is differentiable at z.

[10] 2. Show that the function $u(x, y) = e^x \sin y$ is harmonic and find all harmonic conjugates $v(x, y)$. Then write $f(z) = u(x, y) + iv(x, y)$ as a function of z.

[10] 3. Let $f(z) = \dfrac{1}{z^3 - z^4}$.

(a) Find the Laurent series expansion of $f(z)$ about 0, valid for $0 < |z| < 1$.

(b) Find the Laurent series expansion of $f(z)$ about 0, valid for $|z| > 1$.

[30] 4. Evaluate and simplify the following integrals, using complex variable techniques:

(a) $\int_C \overline{z} dz$, where C is the straight line in the complex plane from the point 2 to the point i.

(b) $\displaystyle\int_{|z|=2} \frac{e^{2z} + z}{z - 1}\, dz$

(c) $\displaystyle\int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4\cos\theta}\, d\theta.$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 16)^2}\, dx.$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - x}\, dx.$

[6] 5. Suppose that $\lim_{n\to\infty} a_n = L$, where (a_n) is a sequence of complex numbers and L is a complex number. Use the ε, N definition of the limit of a sequence to prove that

$$\lim_{n\to\infty} (a_n + 421i) = L + 421i.$$

[8] 6. (a) State Liouville's Theorem.

(b) Do **ONE** of the following two questions: (i) use Cauchy's integral formula for derivatives and the ML Lemma to prove Liouville's Theorem; (ii) use Liouville's Theorem to prove the Fundamental Theorem of Algebra.

[5] 7. Give the definition of Log z for z a nonzero complex number. Then use it to find a negative real number z so that $(\text{Log } z)^3$ is real.

$\overline{[75]}$