Exercise 1.1.1 Let $z_1 = 2 + i$, $z_2 = 3 - 4i$ and $z_3 = 7 + 5i$. Find

(i) $z_1 - 2z_2$

(ii) $z_1 z_3 + z_2$

(iii) z_2^3

(iv) $\dfrac{z_1}{z_3}$

(v) $\dfrac{z_1 z_2}{z_1 + \overline{z}_3}$

Exercise 1.1.2 Give a simple example to show that if $z \in \mathbb{C}$, $|z|^2 \neq z^2$ in general.

Exercise 1.1.3 Find the values of

(i) $\displaystyle\sum_{n=0}^{11} i^n$

(ii) $\left|\dfrac{1}{1+3i} - \dfrac{1}{1-3i}\right|$

Exercise 1.1.4 Let

$$z = \frac{1 - \cos\theta + i\sin\theta}{1 + \cos\theta - i\sin\theta}$$

where θ is real. Show that $\operatorname{Re} z = 0$ and $\operatorname{Im} z = \tan(\theta/2)$.

Exercise 1.1.5 Find the real numbers a and b such that $z = 1 + 2i$ is a solution to the cubic equation

$$z^3 + az + b = 0$$

and find all other solutions in this case.

Exercise 1.1.6 Prove that $z_1 z_2 = 0$ if and only if $z_1 = 0$ or $z_2 = 0$.

Exercise 1.1.7 Find the roots of the equation $z^3 = 1$. If $z^3 = 1$ with $z \neq 1$, show that $z^2 + z^4 = -1$.

Exercise 1.1.8 Find the values of $z \in \mathbb{C}$ which satisfy $|z - i| \leqslant |z - 2|$.

Exercise 1.1.9 Show that for $\alpha, \beta \in \mathbb{R}$ with $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$, $\alpha^2 + \beta^2 \leqslant 1 + \alpha^2\beta^2$. Hence prove that for $a, z \in \mathbb{C}$, with $|a| \leqslant 1$ and $|z| \leqslant 1$,

$$\left|\frac{z - a}{1 - \overline{a}z}\right| \leqslant 1$$

Exercise 1.2.1 Sketch the set of values of z in the complex plane for which

(i) $|z - 2| < 2$ and $|z - i| > 2$

(ii) $\operatorname{Re} z - \operatorname{Im} z < 1$

(iii) $\left|\dfrac{z + 2i}{z - 2i}\right| \leqslant 3$

Exercise 1.2.2 Prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

for all $z_1, z_2 \in \mathbb{C}$. What is the geometrical interpretation of this result?

Exercise 1.2.3 Use a geometrical argument to support the fact that

$$|z - 3i| + |z + 3i| = 12$$

is the equation of an ellipse in the complex plane. Prove this fact using an algebraic approach.

Exercise 1.2.4 Show that the equation

$$(\overline{z} + z)^2 = 2(\alpha\overline{z} + \overline{\alpha}z) \qquad (\alpha \in \mathbb{C})$$

represents a parabola in the complex plane if α is not real. What curve(s) does it represent if α is real?

Exercise 1.2.5 Which of the following sets of complex numbers are (a) open, (b) closed, (c) connected, (d) simply connected, (e) bounded, (f) compact? Give brief reasons for your answers.

(i) $\{z : \operatorname{Im} z < 0\}$

(ii) $\{z : |z - \alpha| \geqslant 4\}$

(iii) $\{1, i, -1, -i\}$

(iv) $\{z : |z| < 1\} \cup \{z : |z - 2i| < 1\}$

(v) $\{z : 1 \leqslant |\operatorname{Re} z| + |\operatorname{Im} z| < 4\}$

(vi) $\mathbb{C}$

Exercise 1.2.6 Prove that $S \subseteq \mathbb{C}$ is closed if $\mathbb{C} \backslash S$ is open.

Exercise 1.2.7 Find the image of the circle with equation

$$(x - a)^2 + (y - b)^2 = r^2 \qquad (a, b, r \in \mathbb{R})$$

under stereographic projection.

Exercise 1.3.1 Find the polar representation of $\sin\theta - i\cos\theta, \theta \in \mathbb{R}$.

Exercise 1.3.2 Express the following in the form $a + ib$ by first finding the polar representation of each of the complex numbers involved:

(i) $(1 + i)^{1000}$

(ii) $(1 - i)^8 (1 + i\sqrt{3})^3$

(iii) $\dfrac{(\sqrt{3} - i)^3}{(-1 + i\sqrt{3})^5}$

(iv) $27^{1/3}\, i^{-1/2}$

(v) $(-1)^{1/8}$

(vi) $(\sqrt{3} + i)^{1/4}$

Exercise 1.3.3 Find the two square roots of $3 + 4i$ in the form $a + ib$. Hence solve the equation

$$3z^2 + (2 + 7i)z + (2i - 4) = 0$$

Exercise *1.3.4 Use polar representation to prove that every complex number $z \neq -1$ of unit modulus can be expressed as

$$z = \frac{1 + it}{1 - it} \qquad (\text{for some } t \in \mathbb{R})$$

Exercise 1.3.5 Use De Moivre's theorem to prove the following identities in real numbers. (You may assume that $\sin^2\theta + \cos^2\theta = 1$.)

(i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(iii) $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$

(iv) $\cos^8\theta + \sin^8\theta = \frac{1}{64}\cos 8\theta + \frac{7}{16}\cos 4\theta + \frac{35}{64}$

Exercise 1.3.6

(i) Find the sixth roots of 1 and prove that they are the vertices of a regular hexagon, centre 0, in the complex plane.

*(ii) Prove that

$$\sin(\pi/n)\sin(2\pi/n)\ldots\sin((n-1)\pi/n) = n2^{1-n} \qquad (n \in \mathbb{N}, n \neq 1)$$

(*Hint.* Find the product of the non-zero roots of $(1 - z)^n = 1$.)