Pmat 421 Winter 08
Assignment # 1 Solution

1. $\left(\frac{i}{1-i}+\frac{1-i}{i}\right)^3=\left(\frac{i\,(1+i)}{1+1}-\frac{(1-i)\,i}{1}\right)^3=\left(\frac{i-1-2-2i}{2}\right)^3=$
$=\frac{-1}{8}\,(3+i)\,(3+i)^2=\frac{-1}{8}\,(3+i)\,2\,(4+3i)=-\frac{1}{4}\,(9+13i)\,.$

2. Compare real and imaginery parts:
$\bar{z}^2=-\left|z\right|^2 \Leftrightarrow \quad x^2-y^2-i2xy=-x^2-y^2 \Leftrightarrow x^2=-x^2$ and $xy=0$
together $x=0, y$ any or $z=iy, y$ real.

3. the set (a) $|z+i|\leq 2$ closed circular disk with centre $-i$,radius $\sqrt{2}$;
(b) $z^2+(\bar{z})^2=x^2-y^2+i2xy+x^2-y^2-i2xy=2\to x^2-y^2=1$
hyperbola with intercepts $x=\pm 1, y=0$.

4. First $(-1-i)^8=\left(\sqrt{2}e^{-i\frac{3}{4}\pi}\right)^8=2^4e^{-i\frac{3}{4}\pi\cdot 8}=16e^{-i6\pi}=16$ since $\theta=\arctan 1-\pi$
then $\left(1+i\sqrt{3}\right)^4=\left(2e^{i\frac{\pi}{3}}\right)^4=2^4e^{i\frac{4}{3}\pi}=16e^{i\pi}e^{i\frac{\pi}{3}}=-16e^{i\frac{\pi}{3}}$since $\theta=\arctan\sqrt{3}$
together $(-1-i)^8\left(1+i\sqrt{3}\right)^4=-16^2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=-128\left(1+i\sqrt{3}\right).$

5. For $z=\left(\frac{1+i}{1-i}\right)^3=\left(\frac{(1+i)^2}{2}\right)^3=\left(\frac{2i}{2}\right)^3=-i$ OR
$\left(\frac{1+i}{1-i}\right)^3=\left(\frac{\sqrt{2}e^{i\frac{\pi}{4}}}{\sqrt{2}e^{-i\frac{\pi}{4}}}\right)^3=e^{i\frac{3}{2}\pi}$ $\quad \arg z=\frac{3}{2}\pi+2k\pi$
and $Arg\ z=-\frac{\pi}{2}(k=-1)$ then both foots$\sqrt{-i}=\left(e^{i\frac{-1}{2}\pi+i2k\pi}\right)^{\frac{1}{2}}=e^{i\pi\left(-\frac{1}{4}+k\right)}$
for $k=0$ $\quad e^{-i\frac{\pi}{4}}=\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$ for $k=1$ $\quad -e^{-i\frac{\pi}{4}}=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$
$\sqrt{-i}=\pm\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$

6. generally for $z\neq 0$ $\;Arg\frac{1}{z}=-Arg\,(z)+2k\pi$ for some k
if $Arg\,(z)\in(-\pi,-\pi)$ then also $-Arg\,(z)\in(-\pi,\pi)$and $k=0$
Only if $\quad z=x, x<0 \quad Argx=Arg\frac{1}{x}=\pi\neq -Arg\,(x)=-\pi(k=1)$

7. Show that $|z+w| \leq |z|+|w|$ .You may use geometry.
to prove it analytically square both sides and use $z=x+iy, w=a+ib$
x, y, a, b real numbers
$(x+a)^2+(y+b)^2 \leq x^2+y^2+a^2+b^2+2\sqrt{x^2+y^2}\sqrt{a^2+b^2}$
simplify $xa+yb \leq \sqrt{x^2+y^2}\sqrt{a^2+b^2}$
if the left side ie negative or 0- done , if positive square again
$(xa)^2+2xayb+(yb)^2 \leq x^2a^2+x^2b^2+y^2a^2+y^2b^2$
$2xayb \leq x^2b^2+y^2a^2 \qquad 2AB \leq A^2+B^2 \qquad 0 \leq (A-B)^2$
true for any A, B where $A=bx \qquad B=ay$
also
from geom. interpretation in a triangle/patallelogram
corresponding to addition $|z+w|$ length of the diagonal;
$|z|, |w|$ length of two sides
Or
$|z+w|^2=(z+w)(\bar{z}+\overline{w})=z\bar{z}+w\bar{z}+z\overline{w}+w\overline{w}=|z|^2+w\bar{z}+\overline{(w\bar{z})}+|w|^2=$
$=|z|^2+2\operatorname{Re}(w\bar{z})+|w|^2 \leq |z|^2+2|w\bar{z}|+|w|^2 \leq |z|^2+2|w||z|+|w|^2=$
$(|z|+|w|)^2$

8. Polar form of $-1=e^{i\pi(1+2k)} \qquad (-1)^{\frac{1}{4}}=e^{i\pi\frac{(1+2k)}{4}}=e^{i\pi\left(\frac{1}{4}+\frac{k}{2}\right)}$
for $k=0 \qquad z_1=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}} \qquad k=1 \qquad z_2=e^{i\pi\frac{3}{4}}=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$
$k=2 \qquad z_3=e^{i\pi\frac{1}{4}}e^{i\pi}=-z_1=-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$
$k=3 \qquad z_4=e^{i\pi\frac{3}{4}}e^{i\pi}=-z_2=\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$
also from the symmetry if we have z_1 the other roots are: $-z_1$,
and $\pm\ \bar{z}_1$the angles between roots are $\frac{\pi}{2}$
De Moivre's Theorem: $R.S. \qquad e^{i\theta 4}=\cos(4\theta)+i\sin(4\theta)$
$L.S.\ \left(e^{i\theta}\right)^4=(\cos\theta+i\sin\theta)^4=\sum_{k=0}^{k=4}\binom{4}{k}\cos^{4-k}\theta\cdot i^k\sin^k\theta$
we need to compare real parts i.e. only $k=0,2,4$.
$i^0=1 \qquad i^2=-1 \qquad i^4=1 \qquad \binom{4}{0}=1, \binom{4}{2}=6, \binom{4}{4}=1$
$\cos(4\theta)=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$

9. Sketch the set $\{\operatorname{Im}(z^2)>1\}=\{(x,y)\,;2xy>1\}$
two separate parts for $x>0 \qquad y>\frac{1}{2x}>0$ above the hyperbola
for $x<0 \qquad y<\frac{1}{2x}<0$ below the hyperbola $y=\frac{1}{2x}$
the boundary is the hyperbola $y=\frac{1}{2x}$ not included
. the set opne,unbounded and not connected.