Pmat 421
Assignment # 2 -Solution

1. Express $(-\sqrt{3-i})^{99}$ in the form of $a+ib, a, b$ real.
First polar form: $-\sqrt{3}-i = 2e^{-i\frac{5}{6}\pi}$ since $\theta = \arctan\frac{1}{\sqrt{3}} - \pi$
then $(-\sqrt{3-i})^{99} = 2^{99}e^{-i\pi\left(\frac{5}{6}\cdot 99\right)} = 2^{99}e^{-i\frac{\pi}{2}} = -i2^{99}$
since $\frac{5}{6}\cdot 99 = \frac{5}{2}\cdot 33 = \frac{165}{2} = \frac{164}{2} + \frac{1}{2} = 82 + \frac{1}{2}$ and $e^{-i82\pi} = 1$

2. Solve $z^2 = 7 - 24i$ in the form of $a+ib, a, b$ real
$(x+iy)^2 = x^2 - y^2 + 2ixy = 7 - 24i \qquad x^2 - y^2 = 7$ and $xy = -12$
solve the system of equations
$y = -\dfrac{12}{x} \to x^2 - \frac{144}{x^2} = 7 \to x^4 - 7x^2 - 144 = \left(x^2 - 16\right)\left(x^2 + 9\right)$
thus $x = \pm 4 \qquad y = \mp 3 \qquad z = \pm(4 - 3i)$.

3. For $w = \frac{1}{z}$ find the image of the set $S = \{z; |z| = 2, \operatorname{Im} z > 0\}$.
using polar form $z = re^{i\theta}$
$S = \{z; r = 2, \theta \in (0,\pi)\}$ and $w = \frac{1}{r}e^{-i\theta}$
$f(S) = \left\{|w| = \frac{1}{2}, \arg w \in (-\pi, 0)\right\}$

4. What is the best upper bund of $|z-3|$ if $z \in N_1(i)-$ the neighborhood of i with radius 1?
$|z-3| = |z-i+i-3| \le |z-i| + |i-3| \le 1 + \sqrt{10}$.

5. For $f(z) = e^{\frac{1}{z}}$ find the domain of definition and the functions u and v such that $f(z) = u(x,y) + iv(x,y)$ for $z = x+iy$.Is it onto C ? Explain..
for $z \neq 0 \qquad \dfrac{1}{\bar{z}} = \dfrac{x+iy}{x^2+y^2} \qquad e^{\frac{1}{z}} = e^{\operatorname{Re}\frac{1}{z}}e^{i\operatorname{Im}\frac{1}{\bar{z}z}}$
$u(x,y) = e^{\frac{x}{x^2+y^2}}\cos\frac{y}{x^2+y^2} \qquad v(x,y) = e^{\frac{x}{x^2+y^2}}\sin\frac{y}{x^2+y^2}$
we know that the range of exp. f. is not including 0 so it is not onto;
generally,two numbers are missing from the range of f: 0 and 1
since $e^{\#}$ is never zero and $e^0 = 1$ but $\frac{1}{z}$ is never 0.

6. Sketch/describe the set $\left|e^{z-\frac{1}{z}}\right| = 1$.Is it open,closed, bounded,connected? Explain.
for $z \neq 0 \qquad \left|e^{z-\frac{1}{z}}\right| = e^{\operatorname{Re}\left(z-\frac{1}{z}\right)} = 1$ iff $\operatorname{Re}(z - \frac{1}{z}) = 0$
$z - \frac{1}{z} = x - \dfrac{x-iy}{x^2+y^2} \to x - \dfrac{x}{x^2+y^2} = x\left(1 - \dfrac{1}{x^2+y^2}\right) = 0$

so $x=0$ or $x^2+y^2=1$ $z=iy, y\neq 0$ or $|z|=1$

unit circle and y-axis without the origin

the set is connected , unbounded and closed since the boundary = the set.

7. $\lim_{z\to i}\frac{z^2+iz+2}{3+4iz-z^2}=\lim_{z\to i}\frac{(z-i)(z+2i)}{-(z-i)(z-3i)}=\lim_{z\to i}\frac{(z+2i)}{-(z-3i)}=\frac{3i}{2i}=\frac{3}{2}.$

8. for a) $\lim_{z\to 0}\frac{iz+\bar{z}}{|z|^2}$:

   if $z=x(y=0)$ $\frac{iz+\bar{z}}{|z|^2}=\frac{x(1+i)}{|x|^2}=\frac{1+i}{x}$ $\left|\frac{1+i}{x}\right|\to\infty$

   also $z=iy\,(x=0)$ $\frac{iz+\bar{z}}{|z|^2}=\frac{-y\,(1+i)}{y^2}$ $\left|\frac{1+i}{-y}\right|\to\infty$

   but $z=x\,(1+i)\,(y=x)$ $\frac{iz+\bar{z}}{|z|^2}=0$ thus limit DNE-does not exist.

   for b) $\lim_{z\to\infty}\frac{iz+\bar{z}}{|z|^2}=0$ since $\left|\frac{iz+\bar{z}}{|z|^2}\right|\leq\frac{|iz|+|\bar{z}|}{|z|^2}=\frac{2\,|z|}{|z|^2}=\frac{2}{|z|}\to 0$ as $|z|\to\infty$

9. Find all z where $f(z)=z^2\bar{z}$ is differentiable, then find $f'(z)$ for such z.

   $f(z)=z^2\bar{z}$ could be differentiable only at 0 since $z^2$is differentiable everywhere and $\bar{z}$ nowhere

   and

   $f\prime(0)=\lim_{z\to 0}\frac{z^2\bar{z}-0}{z}=\lim_{z\to 0}z\bar{z}=0$

   OR

   $f(z)=z^2\bar{z}=\left(x^2-y^2+2ixy\right)(x-iy)\to u=x^3+xy^2$ $v=x^2y+y^3$

   check C.R.$u_x=3x^2+y^2=v_y=x^2+3y^2$ and $u_y=2xy=-v_x=-2xy$

   it must $x^2=y^2$ and $xy=0\to\to(0,0)$

   partials are cont evrywhere , C.R. are satisfied only for $z=0$ thus f has derivative

   only at $z=0$ and $f'(0)=u_x\,(0,0)+iv_x\,(0,0)=0$

   Nowhere analytic

10. For $f(z)=\frac{z^2}{\bar{z}}$ for $z\neq 0$ and $f(0)=0$. f is continous but NOT differntiable at 0

    $f'(0)=\lim_{z\to 0}\frac{\frac{z^2}{\bar{z}}-0}{z}=\lim_{z\to 0}\frac{z}{\bar{z}}$ DNE since $\frac{z}{\bar{z}}=1$ for $z=x$ and $\frac{z}{\bar{z}}=-1$ for $z=iy$