PMAT 421 WINTER 08
Assignment #4 Due by Friday April 11, 4pm:

1. since $z=-1$ is a pole of order 10 of $f(z)=\dfrac{z^{12}}{(z+1)^{10}}$

   $\operatorname{Re}s_{-1}f(z)=\dfrac{1}{9!}\left[(z+1)^{10}f(z)\right]^{(9)}=\dfrac{1}{9!}\left[z^{12}\right]^{(9)}(\text{ at } z=-1)=\dfrac{12\cdot 11\cdot .10\cdot 9..4}{9!}z^4=$
   $=\dfrac{12\cdot 11\cdot .10\cdot 9..4\cdot 3\cdot 2\cdot 1}{9!3!}(-1)^4=\dfrac{12\cdot 11\cdot .10}{3\cdot 2}=220.$

2. The integrand function is analytic except the branch cut $\{\operatorname{Re} z\leq 0, \operatorname{Im} z=0\}$

   and where $\sin(z-1)=0 \qquad z-1=k\pi \qquad z=1+k\pi$ for an integer k

   only two of those are inside the given curve $z_0=1$ and $z_1=1+\pi=4.14...$

   e.g.$z_2=1+2\pi>7 \qquad z_{-1}=1-\pi$ negative...

   we need Residues :but $z=1$ is a removable singularity since

   $\lim\limits_{z\to 1}\dfrac{Log\ z}{\sin(z-1)}=\lim\limits_{z\to 1}\dfrac{\frac{1}{z}}{\cos(z-1)}=1\ (L.H.rule)$ thus Residue is 0;

   $z=1+\pi$ is a simple pole thus

   $\operatorname{Re}s_{1+\pi}f(z)=\lim\limits_{z\to 1+\pi}\dfrac{(z-1-\pi)\,Log\ z}{\sin(z-1)}=$

   $=Log\,(1+\pi)\cdot(\text{L.H.rule})\lim\limits_{z\to 1+\pi}\dfrac{1}{\cos(z-1)}=-\ln(1+\pi)$ and

   $\displaystyle\int_c\frac{Log\ z}{\sin(z-1)}dz=2\pi i\operatorname{Re}s_{1+\pi}f(z)=-2\pi\ln(1+\pi)i.$

3. since $z=0$ is an essential singularity of $f(z)=z^3\cos\dfrac{2}{z}$ use Laurent series

   $\cos w=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}w^{2n} \qquad f(z)=z^3\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n}}{(2n)!}\left(\frac{1}{z}\right)^{2n}=z^3\left(1-\frac{2}{z^2}+\frac{2^4}{4!}\cdot\frac{1}{z^4}-....\right)$

   thus $\operatorname{Re}s_0f(z)=\dfrac{2^4}{4!}=\dfrac{2}{3}.$

4. For all singular points solve $\qquad e^{2z}+e^z(1-i)-i=0$

   first $\qquad w=e^z \qquad w^2+w(1-i)-i=(w-i)(w+1)=0$

   so $w_1=i$ and $w_2=-1 \qquad e^z=i$ and $e^z=-1$

   two groups $\qquad z_k=\log i=\dfrac{\pi}{2}i+2k\pi i \qquad z_n=\log(-1)=i\pi(1+2n)$

   where k and n are any integers;all are simple poles

   $\operatorname{Re}s_{z_k}\ f(z)=\left[\dfrac{1}{(e^{2z}+e^z(1-i)-i)'}\right]_{z_k}=$

$$\left[\frac{1}{2e^{2z}+e^{z}(1-i)}\right]_{z_k}=\frac{1}{2(i)^2+i(1-i)}=\frac{1}{-1+i}=\frac{-1-i}{2}$$

$$\operatorname{Re}s_{z_n}\ f(z)=\left[\frac{1}{(e^{2z}+e^{z}(1-i)-i)'}\right]_{zn}=$$

$$\left[\frac{1}{2e^{2z}+e^{z}(1-i)}\right]_{z_n}=\frac{1}{2(-1)^2-1(1-i)}=\frac{1}{1+i}=\frac{1-i}{2}.$$

5. First all singular points of the integrand function $\quad e^{z}=-e^{-z} \quad e^{2z}=-1$

so $2z=\log(-1)=\pi i+2\pi k i \qquad z_k=\frac{\pi}{2}i+\pi k i \qquad k$ any integer

$z_0=\frac{\pi}{2}i$ inside $\qquad z_1=\frac{3}{2}\pi i$ inside

all others outside e.g. $z_{-1}=-\frac{\pi}{2}i \qquad z_2=\frac{5}{2}\pi i$.. thus

$$\int_c \frac{z}{e^{z}+e^{-z}}dz=2\pi i\left[\operatorname{Re}s_{z_0}+\operatorname{Re}s_{z_1}\right]=2\pi i\left[\frac{\frac{\pi}{2}i}{2i\sin\frac{\pi}{2}}+\frac{\frac{3\pi}{2}i}{2i\sin\frac{3\pi}{2}}\right]=$$

$=2\pi i\left[\frac{\pi}{4}-\frac{3}{4}\pi\right]=-\pi^2 i \qquad$ since all are simple poles and

$$\operatorname{Re}s_{z_z}=\lim_{z\to z_k}\frac{z(z-z_k)}{e^{z}+e^{-z}}=z_k\lim_{z\to z_k}\frac{1}{e^{z}-e^{-z}}=\frac{z_k}{2i\sin z_k}.$$

6. Evaluate $\int_{-1}^{1} z^{i}dz=F(1)-F(-1)$ where $F(z)=\frac{z^{1+i}}{1+i}$

we know that $z^{1+i}=e^{(1+i)\log z}$ where for $\log z$ we are using the branch with $\arg z\in\left(\frac{\pi}{2},\frac{5}{2}\pi\right)$

so $\log 1=0+2\pi i \qquad \log(-1)=0+\pi i$

$F(1)=e^{2\pi i(1+i)}=e^{2\pi i}e^{-2\pi}=e^{-2\pi} \qquad F(-1)=e^{\pi i(1+i)}=e^{\pi i}e^{-\pi}=-e^{-\pi}$

thus the integral $I=\frac{1}{1+i}\left(e^{-2\pi}+e^{-\pi}\right)=\frac{1-i}{2}\left(e^{-2\pi}+e^{-\pi}\right).$

7. For singular points of $f(z)=\frac{1}{e^{z^2}-1}$ solve $e^{z^2}=1 \qquad z^2=\log 1=2k\pi i$

$k=0 \qquad z_0=0 \qquad$ double pole since $\lim\limits_{z\to 0}\frac{z^2}{e^{z^2}-1}=\lim\limits_{z\to 0}\frac{2z}{e^{z^2}2z}=1\neq 0$

$k>0 \qquad z^2=2k\pi e^{i\frac{\pi}{2}} \qquad z_k=\pm\sqrt{2k\pi}e^{i\frac{\pi}{4}}=\pm\sqrt{k\pi}(1+i)$

$k<0 \qquad z^2=2|k|\pi e^{-i\frac{\pi}{2}} \qquad z_k=\pm\sqrt{2|k|\pi}e^{-i\frac{\pi}{4}}=\pm\sqrt{|k|\pi}(1-i)$

for $k\neq 0$ $\qquad$ simple poles so

$\operatorname{Re}s_{z_k}f(z)=\left[\frac{1}{e^{z^2}2z}\right]_{z_k}=\frac{1}{2z_k}=\frac{\pm(1-i)}{2\sqrt{k\pi}}$ for $k>0$

and $\operatorname{Re}s_{z_k}f(z) = \dfrac{1}{2z_k} = \dfrac{\pm(1+i)}{2\sqrt{|k|\,\pi}}$ for $k<0$

BUT

$\operatorname{Re}s_0 f(z) = \lim\limits_{z\to 0}\left[\dfrac{z^2}{e^{z^2}-1}\right]' = \lim\limits_{z\to 0}\dfrac{2z\left[e^{z^2}-1-z^2e^{z^2}\right]}{(e^{z^2}-1)^2} = \lim\limits_{z\to 0}2z\cdot\lim\limits_{w\to 0}\dfrac{e^w-1-we^w}{(e^w-1)^2} = 0$

since for the second one we can use L'H.Rule

$\lim\limits_{w\to 0}\dfrac{e^w-1-we^w}{(e^w-1)^2} = \lim\limits_{w\to 0}\dfrac{e^w-e^w-we^w}{2(e^w-1)e^w} = \lim\limits_{w\to 0}\dfrac{-w}{2(e^w-1)} = -\dfrac{1}{2}$

also in Laurent series we have only even powers so $b_1 = 0$

8. First $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2+1}{x^4+1}dx$

we create a curve γ consisting from the line segment $[-R,R]$ and c_R =upper half of the circle

$|z| = R > 1$ oriented positively

then $\displaystyle\int_\gamma \frac{z^2+1}{z^4+1}dz = 2\pi i\,[\,Res_{z_1} + \operatorname{Re}s_{z_2}]$ where z_1 and z_2 are poles with $\operatorname{Im} z > 0$

solve $\quad z^4 = -1 = e^{i\pi+2k\pi i} \quad z_k = e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}$

for $k=0 \quad z_1 = \dfrac{1}{\sqrt{2}}(1+i) \quad z_1^2 = i$

$k=1 \quad z_2 = \dfrac{1}{\sqrt{2}}(-1+i) \quad z_2^2 = -i$ (the other two $z = \dfrac{1}{\sqrt{2}}(\pm 1 - i)$are outside γ)

$\operatorname{Re}s_{z_k} = \dfrac{z_k^2+1}{4z_k^3} = \dfrac{z_k^2+1}{4z_k^4}\cdot z_k \quad z_k^4 = -1$

$Res_{z_1} + \operatorname{Re}s_{z_2} = \dfrac{i+1}{-4}\cdot\dfrac{1}{\sqrt{2}}(1+i) + \dfrac{-i+1}{-4}\cdot\dfrac{1}{\sqrt{2}}(-1+i) = \dfrac{i}{-2\sqrt{2}}\cdot 2 = -\dfrac{i}{\sqrt{2}}$

since the degree of the bottom $\geq$ than the degree of the top we know that

$\displaystyle\int_{c_R}\frac{z^2+1}{z^4+1}dz \to 0$ as $R \to 0$

$$\int_{c_R}\frac{z^2+1}{z^4+1}dz + \int_{-R}^{R}\frac{x^2+1}{x^4+1}dx = \int_\gamma\frac{z^2+1}{z^4+1}dz = 2\pi i\,[\,Res_{z_1} + \operatorname{Re}s_{z_2}] = \sqrt{2}\pi$$

and finally $\displaystyle\int_0^\infty\frac{x^2+1}{x^4+1}dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2.}}$

TOTAL out of 50: