PMAT 421 WINTER 99
FINAL 3 hours

1. Find all values of $\left(-\frac{\pi}{2}\right)^{\frac{1}{i}}$ in the form $a+ib$ where a, b are real numbers. [5]

2. Find all solutions of $\sin z = 2$ in the form $a+ib$ where a, b are real numbers. [9]

3. Explain Bernoulli's paradox:
Since $(-z)^2 = z^2$ then for $z \neq 0$ $Log(-z)^2 = Log\ z^2$ and
$2Log\ (-z) = 2\ Log\ z$ and by cancelling 2 $Log(-z) = Log\ z$.
Find the error in the reasoning! [5]

4. Find the Laurent series of $f(z) = \dfrac{1}{(z-4)^2}$ around $z_0 = 1$
in the domain containing the point -4. [9]

5. Explain why for any $R > 0$ there are infinitely many z such that
$|z| > R$ and $|\cos z| > 100$.
State the theorem used. [7]

6. Evaluate $\int_c \operatorname{Im} z\ dz$ where c is a part of the circle centered at i with radius 1
between the origin and $1+i$. [9]

7. For $f(z) = \dfrac{1}{z \sin z}$
   (a) classify all singularities;
   (b) find the residue at $z_0 = 0$. [9]

8. Evaluate $\int_0^\infty \dfrac{x^2}{x^4+4} dx$ by means of Residue Theorem.Explain all your steps. [9]

9. Evaluate $\int_0^{2\pi} \dfrac{\cos 2\theta}{5+4\cos\theta} d\theta$ by means of Residue Theorem.Explain all your steps. [9]

10. For $w = \log z$ with $\frac{\pi}{4} \leq \arg z < \dfrac{9\pi}{4}$ find
   (a) where the mapping is conformal;
   (b) the image of the y axis without the origin;
   (c) the image of the unit circle in the w plane. [9]

SOLUTIONS

For 1)

$\left(-\frac{\pi}{2}\right)^{\frac{1}{i}} = e^{\frac{1}{i}\log\left(\frac{-\pi}{2}\right)} = e^{\frac{1}{i}\left[\ln\frac{\pi}{2}+i(\pi+2k\pi)\right]} = e^{-i\ln\frac{\pi}{2}} \cdot e^{\pi(2k+1)} =$

$= e^{\pi(2k+1)}\cos\left(\ln\frac{\pi}{2}\right) - ie^{\pi(2k+1)}\sin\left(\ln\frac{\pi}{2}\right)$ for any integer k.

For 2)

we can use the definition of sin : $\dfrac{e^{iz}-e^{-iz}}{2i} = 2$ and $w = e^{iz}$

we get $w - \frac{1}{w} = 4i$ and then $\quad w^2 - 4iw - 1 = 0$

$w_{1/2} = \frac{4i\pm\sqrt{-16+4}}{2} = i\left(2\pm\sqrt{3}\right)$ or $w_1 = i\left(2+\sqrt{3}\right), w_2 = \dfrac{i}{2+\sqrt{3}}$

now

$z = \frac{1}{i}\log w = -i\left[\ln\left(2\pm\sqrt{3}\right) + i\left(\frac{\pi}{2}+2k\pi\right)\right] = \frac{\pi}{2} + 2k\pi \pm i\ln\left(2+\sqrt{3}\right)$

for any integer k.

For 3)

the property of log : $\log w^2 = 2\log w + i2n\pi$ for certain n ONLY

it means that $2\log(-z) = 2\log z + i2n\pi$

in detail if $|Arg\ z| > \frac{\pi}{2}$ i.e. the point z is in the left half

$2\cdot|Arg\ z| > \pi$ so it is not principal value anymore thus in this case

$Arg\ z^2 \neq 2\cdot Arg\ z$.

For 4)

the center is 1 and singular point is 4 so we have two possible domains

$|z-1| < 3$ or $|z-1| > 3$ we need the latter so negative powers of $(z-1)$:

first

$$\frac{1}{z-4} = \frac{1}{(z-1)-3} = \frac{1}{z-1}\cdot\frac{1}{1-\frac{3}{z-1}} = \left(\text{for } \left|\frac{3}{z-1}\right| < 1\right)$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{3^n}{(z-1)^n} = \sum_{n=0}^{\infty}\frac{3^n}{(z-1)^{n+1}} = \sum_{k=1}^{\infty}3^{k-1}(z-1)^{-k}$$

differentiate

$$\frac{-1}{(z-4)^2} = \sum_{k=1}^{\infty}3^{k-1}(-k)(z-1)^{-k-1}$$

and for $|z-1| > 3$

$$\frac{1}{(z-4)^2} = \sum_{k=1}^{\infty}3^{k-1}(k)(z-1)^{-k-1} = \sum_{k=1}^{\infty}\frac{k3^{k-1}}{(z-1)^{k+1}} = \sum_{n=2}^{\infty}\frac{(n-1)\,3^{n-2}}{(z-1)^n}.$$

For 5)

the function cos is $2\pi-$periodic so if we can find one z we have infinitely many.

By contradiction:

if there is no such z it means there is $R_0 > 0$ that $|\cos z| \le 100$ for all $|z| > R_0$

but cos is continuous for $|z| \le R_0$ so $|\cos z| \le M$ for some $M > 0$

together

cos is entire and bounded so by Liouville's Th. cos $z = \mathit{const}$. which is NOT true.

Liouville's Th.:

If f is entire and bounded i.e $f'(z)$ exists for all z and for some $M > 0$ $|f(z)| \le M$ for all z;

then f is constant .

For 6)

since $\operatorname{Im} z$ is nowhere analytic we have to use the definition of the integral

first parametrize the curve $c : |z - i| = 1$

$z(t) = i + e^{it}$, $t \in \left[-\frac{\pi}{2}, 0\right]$, $\operatorname{Im} z(t) = 1 + \sin t$, $dz = ie^{it}dt$

$$\int_c \operatorname{Im} z \; dz = \int_{-\frac{\pi}{2}}^{0} (1 + \sin t)\, i\, (\cos t + i \sin t)\, dt =$$

$$= i \int_{-\frac{\pi}{2}}^{0} [\cos t + \cos t \sin t]\, dt - \int_{-\frac{\pi}{2}}^{0} \left(\sin t + \sin^2 t\right) dt =$$

$$= \left[i \sin t + i\tfrac{\sin^2 t}{2} + \cos t\right]_{-\frac{\pi}{2}}^{0} - \int_{-\frac{\pi}{2}}^{0} \tfrac{1-\cos 2t}{2} dt = 1 + i - \tfrac{i}{2} - \tfrac{1}{2} \cdot \tfrac{\pi}{2} + \left[\tfrac{\sin 2t}{4}\right]_{-\frac{\pi}{2}}^{0} =$$

$$= (1 - \tfrac{\pi}{4}) + \tfrac{1}{2} i.$$

For 7a)

$z \sin z = 0$ for $z = k\pi$ for any integer k

for $k = 0$ $z_0 = 0$ is a pole of order $m = 2$ since $\lim\limits_{z \to 0} z^2 f(z) = \lim\limits_{z \to 0} \dfrac{z}{\sin z} = 1 \neq 0$

for $k \neq 0$ $z_k = k\pi$ are simple poles

since $\lim\limits_{z \to k\pi} (z - k\pi)\, f(z) = \frac{1}{k\pi} \cdot \lim\limits_{z \to 0} \dfrac{z - k\pi}{\sin z} = \dfrac{1}{k\pi \cos k\pi} = \dfrac{(-1)^k}{k\pi} \neq 0$

for b)

$$\operatorname{Res}[f, 0] = \lim_{z \to 0} \left[z^2 f(z)\right]' = \lim_{z \to 0} \left[\frac{z}{\sin z}\right]' = \lim_{z \to 0} \frac{\sin z - z \cos z}{\sin^2 z} (L'H) =$$

$$= \lim_{z \to 0} \frac{\cos z - \cos z + z \sin z}{2 \sin z \cos z} = 0.$$

For 8)

$$\int_0^\infty \frac{x^2}{x^4+4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{x^4+4}dx$$

we will use Residue Th. for

$$\int_c \frac{z^2}{z^4+4}dz = \int_{c_1} \frac{z^2}{z^4+4}dz + \int_{c_2} \frac{z^2}{z^4+4}dz = 2\pi i \sum \text{Res at all poles inside } c$$

where $c_1 : [-R, R]$ line segment and $c_2 : |z| = R, \operatorname{Im} z > 0$ half circle , for $R > 2$ we can estimate

$$\left|\int_{c_2} \frac{z^2}{z^4+4}dz\right| \leq \max_{c_2}\left|\tfrac{z^2}{z^4+4}\right| \cdot \text{length } c_2 \leq \max_{|z|=R}\frac{|z|^2}{|z|^4-4}\cdot \pi R \leq \frac{\pi R^3}{R^4-4} \to 0$$

as $R \to \infty$

Next, find all singularities of the integrand function i.e. solve $z^4 = -4$

$z^2 = \pm 2i$ and $z = \pm(1+i)$ or $\pm(1-i)$ OR $z = 4^{\frac{1}{4}}e^{i\frac{\pi}{4}+i\frac{\pi k}{2}} = \sqrt{2}e^{i\pi\left(\frac{1+2k}{4}\right)}$

inside the curve only poles with $\operatorname{Im} z > 0$ so $z_1 = 1+i, z_2 = -1+i$

$$\operatorname{Res}[f, z_j] = \frac{p(z_j)}{q'(z_j)} = \frac{z_j^2}{4z_j^3} \qquad \text{OR}$$

$$\operatorname{Res}[f, z_j] = \lim_{z\to z_j}(z-z_j)f(z) = \lim_{z\to z_j}(z-z_j)\frac{z^2}{z^4+4} = z_j^2 \cdot \frac{1}{4z_j^3} = \frac{1}{4z_j}$$

$$\operatorname{Re}s[f, z_1] + \operatorname{Re}s[f, z_2] = \tfrac{1}{4}\left[\tfrac{1}{1+i} + \tfrac{1}{-1+i}\right] = -\frac{i}{4}$$

and the integral is $2\pi i \cdot \frac{-i}{4} = \frac{\pi}{2}$ and $\displaystyle\int_0^\infty \frac{x^2}{x^4+4}dx = \frac{\pi}{4}.$

For 9)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}d\theta = \operatorname{Re}\int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta}d\theta = \operatorname{Re}\oint_{|z|=1} \frac{z^2}{5+4\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz} =$$

(by subst. $z = e^{i\theta}, \cos\theta = \frac{1}{2}(z+\frac{1}{z})$ and $dz = ie^{i\theta}d\theta, e^{i2\theta} = z^2$)

$$= \operatorname{Re}\tfrac{1}{i}\oint_{|z|=1} \frac{z^2}{5z+2z^2+2}dz = \operatorname{Re}\frac{1}{i}\oint_{|z|=1} \frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}dz =$$

$$= \operatorname{Re}\left[\tfrac{1}{i}\cdot 2\pi i \cdot \operatorname{Res}\left[f, -\tfrac{1}{2}\right]\right] = \frac{\pi}{6}$$

since only $z = -\frac{1}{2}$ is inside the unit circle and

$$\operatorname{Re}s\left[f, -\tfrac{1}{2}\right] = \left[\tfrac{z^2}{2(z+2)}\right]_{z=\frac{-1}{2}} = \frac{1}{12}.$$

For 10a)

the given branch of log is cont. and therefore analytic in the z -plane except the branch cut : $S = \left\{\theta = \frac{\pi}{4} i.e. \operatorname{Im} z = \operatorname{Re} z \geq 0\right\} = \{y = x, x \geq 0\}$

so mapping is conformal on $C - S$ since $(\log z)' = \frac{1}{z} \neq 0$ there;

For b)

for $x = 0, y > 0$

$w = \ln y + i\frac{\pi}{2}$..........horizontal line since $u = \ln y$ is any real #, $v = \frac{\pi}{2}$

for $x = 0, y < 0$

$w = \ln(-y) + i\left(-\frac{\pi}{2} + 2\pi\right)$ ......horizontal line $v = \frac{3\pi}{2}$

For c)

$|z| = 1$ means $z = e^{i\theta}$ where$\frac{\pi}{4} \leq \theta <\frac{9\pi}{4}$

$w = i\theta$... vert. line segment, $u = 0, \frac{\pi}{4} \leq v <\frac{9\pi}{4}$