

PMAT 421 WINTER 07
FINAL 3 hours

Solution

Note: Give the answers, if possible in the form $a + ib$, a, b real.

Each question is for 10 marks.

1. Find all values of

$$(a) \frac{1}{(-\sqrt{3}-i)^i} = e^{-i \log(-\sqrt{3}-i)} = e^{-i[\ln|\sqrt{3}-i| + i\theta + 2k\pi]} = e^{-i \ln 2} e^{\frac{7}{6}\pi + 2k\pi} = e^{\frac{7}{6}\pi + 2k\pi} [\cos(\ln 2) - i \sin(\ln 2)] \quad (\text{also } \theta = -\frac{5}{6}\pi);$$

$$(b) \arg \left[\frac{(1-i)^{10}}{(1+i)^6} \right] = \arg \left[\frac{(1-i)^{2 \cdot 5}}{(1+i)^{2 \cdot 3}} \right] = \arg \left[\frac{(-2i)^5}{(2i)^3} \right] = \arg(4) = 2k\pi$$

where k is any integer OR

$$\begin{aligned} \arg \left[\frac{(1-i)^{10}}{(1+i)^6} \right] &= \arg(1-i)^{10} - \arg(1+i)^6 + 2n\pi = 10 \cdot \frac{-\pi}{4} - 6 \cdot \frac{\pi}{4} + 2n\pi = \\ &= -\frac{8}{2}\pi + 2n\pi = 2k\pi \end{aligned}$$

2. $\cos z = 3i \rightarrow \frac{e^{iz} + e^{-iz}}{2} = 3i \rightarrow w + \frac{1}{w} = 6i$ where $w = e^{iz}$

$$w^2 - 6iw + 1 = 0 \quad w_1 = \frac{6i + \sqrt{-40}}{2} = i(3 + \sqrt{10})$$

$$\text{and } w_2 = i(3 - \sqrt{10})$$

$$iz = \log i(3 + \sqrt{10}) = \ln(3 + \sqrt{10}) + i\frac{\pi}{2} + 2ki\pi$$

$$z = -i \ln(3 + \sqrt{10}) + \frac{\pi}{2} + 2k\pi \quad \text{also}$$

$$iz = \log i(3 - \sqrt{10}) = \ln(\sqrt{10} - 3) - i\frac{\pi}{2} + 2li\pi$$

$$z = -i \ln(\sqrt{10} - 3) - \frac{\pi}{2} + 2l\pi, \text{ where } k, l \text{ are any integers}$$

$$\text{also } z = -i \ln(\sqrt{10} - 3) - \frac{\pi}{2} + 2l\pi = z = i \ln(\sqrt{10} + 3) - \frac{\pi}{2} + 2l\pi,$$

$$\text{so together } z = \mp i \ln(3 + \sqrt{10}) \pm \frac{\pi}{2} + 2k\pi$$

we can always solve $\cos z = w_0$ for any complex w_0 since

$$\text{a quadratic equation } w^2 - 2iw_0w + 1 = 0$$

has always at least one complex root by Fund.Th. of Algebra,

and $w \neq 0$ so $z = \frac{1}{i} \log w$.

3. For $f(z) = x^2 - y^2 - 2xy + i(x^2 - 2xy)$, where $z = x + iy$, x, y real

$$u(x, y) = x^2 - y^2 - 2xy \quad v(x, y) = (x^2 - 2xy)$$

Necessary condition for f to be differentiable:

$$u_x = 2x - 2y = v_y = -2x \rightarrow 4x = 2y$$

$$u_y = -2y - 2x = -v_x = -2x + 2y \rightarrow y = 0, x = 0$$

necessary conditions: If C.R. are NOT satisfied then f is not differentiable.

Sufficient condition: since partial derivatives are cont. everywhere and

C.R. conditions are satisfied at the origin f is differentiable at the origin only.

But it is analytic nowhere since the origin is not an open set.

4. $u(x, y) = x^3 - 3xy^2 - 2y + 5$

is harmonic since $u_{xx} = (3x^2 - 3y^2)_x = 6x$ and $u_{yy} = (-6xy - 2)_y = -6x$ thus $\Delta u = 0$

For a harmonic conjugate v : $v_y = u_x = 3x^2 - 3y^2$

then $v = \int u_x dy + c(x) = 3x^2 y - y^3 + c(x)$

check $v_x = 6xy + c'(x) = -u_y = 6xy + 2 \rightarrow c(x) = 2x$

together $v = 3x^2 y - y^3 + 2x$ now, substitute $x = \frac{1}{2}(z + \bar{z})$ $y = \frac{1}{2i}(z - \bar{z})$

also we can guess that $u = \operatorname{Re} f(z)$ where $f(z) = z^3 + 2iz + 5$

since $z^3 + 2iz + 5 = (x + iy)^3 + 2i(x + iy) + 5 =$

$= x^3 + 3ix^2y - 3xy^2 - y^3 + i2x - 2y + 5$ then $v = \operatorname{Im} f(z)$.

5. First the Laurent series for $f(z) = \frac{1}{z-1} = \frac{1}{(z-i) + i-1}$ around $z_0 = i$

the distance to the sing point $z = 1$ is $|i-1| = \sqrt{2}$ so we have two possibilities

$|z-i| < \sqrt{2}$ or $|z-i| > \sqrt{2}$ - this one contains 2 since $|i-2| = \sqrt{5}$

then $\frac{1}{z-1} = \frac{1}{(z-i) + i-1} = \frac{1}{(z-i)} \cdot \frac{1}{1 - \frac{1-i}{z-i}} =$

$= \frac{1}{(z-i)} \sum_{n=0}^{\infty} \frac{(1-i)^n}{(z-i)^n} = (k = n+1) = \sum_{k=1}^{\infty} \frac{(1-i)^{k-1}}{(z-i)^k}$, for $|\frac{1-i}{z-i}| < 1$

now differentiate

$$\frac{-1}{(z-1)^2} = \sum_{k=1}^{\infty} \frac{-k(1-i)^{k-1}}{(z-i)^{k+1}} \quad (k+1 = n) \quad \frac{1}{(z-1)^2} = \sum_{n=2}^{\infty} \frac{(n-1)(1-i)^n}{(z-i)^n}$$

and $b_n = (n-1)(1-i)^n$ for $n \geq 2$.

6. Classify all singular points z_k of $f(z) = \frac{\cos z - 1}{z^2(e^z - 1)}$ and then find all $\operatorname{Res}(z_k)$.

first $z = 0$ it is a simple pole since

$$\operatorname{Res}(0) = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \frac{\cos z - 1}{z^2} \left(\frac{0}{0} \text{ L.H.R.} \right) \cdot \lim_{z \rightarrow 0} \frac{z}{e^z - 1} = -\frac{1}{2} \neq 0$$

next.solve: $e^z - 1 = 0 \quad z_k = \log 1 = 2k\pi i, k \neq 0$

$$\begin{aligned} \operatorname{Res}(z_k) &= \lim_{z \rightarrow z_k} (z - z_k) f(z) = \lim_{z \rightarrow z_k} \frac{\cos z - 1}{z^2} \lim_{z \rightarrow z_k} \frac{z - z_k}{e^z - 1} \quad (\text{L.H.R.}) = \\ &= \frac{\cos(2ki\pi) - 1}{-4k^2\pi^2} \frac{1}{e^{2ki\pi}} = \frac{1 - \cosh(2k\pi)}{4k^2\pi^2} \neq 0. \end{aligned}$$

7. (a)

since $f(z) = \frac{1}{z}$ is analytic for $z \neq 0$ and $F(z) = \log z$

the **branch** with $\arg \in [0, 2\pi)$

by **Fund.Th. of Calculus** for any curve from $-1 - i$ to $i\sqrt{2}$ not crossing the branch cut

$$\int_c \frac{1}{z} dz = \log(i\sqrt{2}) - \log(-1 - i) = \ln\sqrt{2} + i\frac{\pi}{2} - [\ln\sqrt{2} + i\frac{5}{4}\pi] = -\frac{3}{4}i\pi.$$

(b) by definition:

$$\int_c \frac{1}{z} dz = - \int_{\frac{\pi}{2}}^{\frac{5}{4}\pi} \frac{1}{\sqrt{2}e^{-it}} \cdot i\sqrt{2}e^{it} dt = \left[\frac{e^{2it}}{2} \right]_{\frac{\pi}{2}}^{\frac{5}{4}\pi} = \frac{1}{2} (e^{\pi i} - e^{\frac{5}{2}\pi i}) = \frac{-1 - i}{2}.$$

since for $-c \quad z(t) = \sqrt{2}e^{it} \quad t \in [\frac{\pi}{2}, \frac{5}{4}\pi]$

from $-1 - i$ to $i\sqrt{2}$ lying in the left half of the complex plane.

8. By L.Th.if $\sin z$ is entire and bounded it must be constant- contradiction.

$$9. \text{ First } I = \int_0^{\infty} \frac{\cos 2x}{x^2(x^2 + 1)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2x}{x^2(x^2 + 1)} dx = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2(x^2 + 1)} dx$$

then $f(z) = \frac{e^{2iz}}{z^2(z^2 + 1)}$ is analytic except at $z = 0$ double pole

and $z = \pm i$ simple poles; the curve c is dented around 0

since the degree of the bottom ≥ 2 + degree of the top we know that

$$\frac{1}{z^2(z^2 + 1)} \rightarrow 0 \text{ as } z \rightarrow \infty \quad \text{thus}$$

$$I = \frac{1}{2} \operatorname{Re}\{2\pi i \operatorname{Res}_i(f) + \pi i \operatorname{Res}_0(f)\}$$

now,

$$\operatorname{Res}(i) = [(z - i) f(z)]_{z=i} = \left[\frac{e^{2iz}}{z^2(z + i)} \right]_{z=i} = \frac{e^{-2}}{-2i}$$

$$\operatorname{Res}(0) = [z^2 f(z)]'_{z=0} = \left[\frac{e^{2iz}}{(z^2 + 1)} \right]'_{z=0} = \left[\frac{2ie^{2iz}}{(z^2 + 1)} - \frac{e^{2iz} \cdot 2z}{(z^2 + 1)^2} \right]_{z=0} = 2i$$

thus

$$I = \frac{1}{2} \operatorname{Re}\{2\pi i \operatorname{Res}(i) + \pi i \operatorname{Res}(0)\} = \frac{-\pi}{2e^2} - \pi.$$

10. For $w = e^{\pi z}$ $w' = \pi e^{\pi z} \neq 0$ for all z so the mapping is conformal for all z ;

but is it NOT one-to-one since $f(z + 2ki) = e^{\pi z} e^{2ki\pi} = e^{\pi z} = f(z)$... $2i$ -periodic

the range is all $w \neq 0$ since we can solve for $z = \frac{1}{\pi} \log w$ if $w \neq 0$

generally, $w = e^{\pi x + \pi y i}$ $u = e^{\pi x} \cos \pi y$ $v = e^{\pi x} \sin \pi y$

the image $y = \frac{1}{2}, x$ any $w = e^{\pi x + \frac{\pi}{2} i}$ $u = 0, v = e^{\pi x} > 0$, upper half of v-axis

the image $y = 1, x$ any $w = e^{\pi x + \pi i}$ $v = 0, v = -e^{\pi x} < 0$, left half of u-axis

a vertical line between $y = y_0 \in \left(\frac{1}{2}, 1\right)$, x any, e.g. $y_0 = \frac{3}{4}$

so $\pi y_0 \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(\pi y_0) > 0$, $\cos(\pi y_0) < 0$

for our example $u = \frac{-1}{\sqrt{2}} e^{\pi x}$ $v = \frac{1}{\sqrt{2}} e^{\pi x}$ so $u = -v, v > 0$

generally

$u = e^{\pi x} \cos \pi y_0, v = e^{\pi x} \sin \pi y_0$ $u = mv, v > 0, m < 0$ a ray in the second quadrant

together the image of the set $\left\{z; \frac{1}{2} \leq \text{Im } z \leq 1\right\}$ the second quadrant of the u, v -plane including axes, excluding the origin