

PMAT 421 WINTER 07
FINAL 3 hours

Note: Give the answers, if possible in the form $a + ib$, a, b real. Each question is for 10 marks.

1. Find all values of

$$(a) \frac{1}{(-\sqrt{3} - i)^i} \quad (b) \operatorname{arg} \left[\frac{(1 - i)^{10}}{(1 + i)^6} \right].$$

2. Solve $\cos z = 3i$. Explain why we can always solve $\cos z = w_0$ for any complex w_0 .

3. For $f(z) = x^2 - y^2 - 2xy + i(x^2 - 2xy)$, where $z = x + iy$, x, y real find where f is differentiable and where analytic.

explain the difference between necessary and sufficient C.R. conditions.

4. Show that $u(x, y) = x^3 - 3xy^2 - 2y + 5$

(a) is harmonic; (b) find a harmonic conjugate v ;

and (c) find $f(z) = u(x, y) + iv(x, y)$ in terms of z .

5. Find the Laurent series for $f(z) = \frac{1}{(z - 1)^2}$ around $z_0 = i$

in the domain containing 2. Describe the domain, find the formula for b_n .

6. Classify all singular points z_k of $f(z) = \frac{\cos z - 1}{z^2(e^z - 1)}$ and then find all $\operatorname{Res}(z_k)$.

7. Evaluate

$$(a) \int_c \frac{1}{z} dz \text{ where } c \text{ is any curve from } -1 - i \text{ to } i\sqrt{2}$$

lying in the left part of the plane;

$$(b) \int_c \frac{1}{\bar{z}} dz \text{ where } c \text{ is the part of the circle from } -1 - i \text{ to } i\sqrt{2}$$

lying in the left half of the complex plane

8. Prove or disprove that $|\sin z| \leq 1$. State the theorem used.

9. Evaluate $\int_0^{\infty} \frac{\cos 2x}{x^2(x^2 + 1)} dx$ by means of the Residue Theorem. Explain all your steps.

10. For $w = e^{\pi z}$ (a) show that the mapping is conformal for all z ; is it *one-to-one*?

(b) find the range; (c) sketch / describe the image of the set $\{z; \frac{1}{2} \leq \operatorname{Im} z \leq 1\}$

in the w -plane.