PMAT 421 WINTER 08
Midterm 2 Time : 90 minutes Solution

1. we know that $I = \int_c \frac{1+z}{z}\, dz = F(1+i) - F(-i)$

   where $F(z) = \log z + z$ but in a domain = complex plane minus brach cut

   for a) we can chose Log z since the curve crossing the x-axis on the right and the pricipal branch cut is on the left then

   $I = Log(1+i) + 1 + i - Log(-i) + i = \ln\sqrt{2} + i\frac{\pi}{4} + 1 + i\frac{\pi}{2} + 2i = 1 + \ln\sqrt{2} + i(2 + \frac{3}{4}\pi)$

   for b) we can chose a branch with $\arg z \in [0, 2\pi)$ then

   $I = \log_0(1+i) + 1 + i - \log_0(-i) + i = \ln\sqrt{2} + i\frac{\pi}{4} + 1 + 2i - i\frac{3\pi}{2} = 1 + \ln\sqrt{2} + i(2 - \frac{5}{4}\pi)$

   Note: the difference is $2\pi i$ since if the curve is closed around the origin $I = 2\pi i$.

2. since the integrand function is NOT analytic we have to evaluate by Definition:

   parametrization $z(t) = 2e^{it}$ $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $z'(t) = 2ie^{it}$ then

   $$\int_c \frac{\operatorname{Re} z}{z}\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos t}{2e^{it}} \cdot 2ie^{it} dt = 2i \cdot 2\left[\sin t\right]_0^{\frac{\pi}{2}} = 4i$$

3. The integrand function is analytic for $z \neq \pm 2i$

   for a)

   none is inside the curve so by Cauchy-G Th. $\oint_c \frac{\sin \pi z}{z^2+4}\, dz = 0;$

   for b)

   both points are in $|z| = 3$ so by Def.Th $\oint_c \frac{\sin \pi z}{z^2+4}\, dz = \oint_{c_1} \frac{\sin \pi z}{z^2+4}\, dz + \oint_{c_2} \frac{\sin \pi z}{z^2+4}\, dz$

   where $c_1 : |z - 2i| = \frac{1}{2}$ and $c_2 : |z + 2i| = \frac{1}{2}$ and then By Cauchy Intgr.formula

   $$\oint_c \frac{\sin \pi z}{z^2+4}\, dz = \oint_{c_1} \frac{\sin \pi z}{z+2i} \cdot \frac{1}{z-2i} dz + \oint_{c_2} \frac{\sin \pi z}{z-2i} \cdot \frac{1}{z+2i}\, dz =$$

   $$= 2\pi i \left[\frac{\sin \pi z}{z+2i}\right]_{z=2i} + 2\pi i \left[\frac{\sin \pi z}{z-2i}\right]_{z=-2i} = 2\pi i \left[\frac{\sin 2\pi i}{4i} - \frac{\sin 2\pi i}{-4i}\right] = \pi \sin 2\pi i =$$

   $$= \pi \frac{e^{-2\pi} - e^{2\pi}}{2i} = i\pi \sinh(2\pi).$$

4. Estimate $\int_c e^{i\bar{z}^2} dz$ where c is the line segment from the origin to $2+i$.

   we know that $\left|\int_c e^{i\bar{z}^2} dz\right| \leq ML$ where $L = |2+i| = \sqrt{5}$

and $M = \max \left| e^{i\bar{z}^2} \right| = \max \left| e^{\mathrm{Re}\, i(x-iy)^2} \right| = \max e^{2xy}$

on the line $y = \dfrac{x}{2}, x \in [0,2]$ $\qquad M = \max e^{x^2} = e^4$

thus the estimate is $\qquad ML = e^4\sqrt{5}.$

5. for a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2iz)^n}{\sqrt{n}}$ use the ratio test for $a_n = \dfrac{(2i)^n}{\sqrt{n}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{2^n} = \sqrt{\frac{n}{n+1}} \cdot 2 \to 2 \text{ so } R = \frac{1}{2}$$

the series is abs.convergent for $|z| < \frac{1}{2}$;

for b) $\displaystyle\sum_{n=1}^{\infty} (ni)^n (z-1)^n$ use the root test for $a_n = (ni)^n$

$\sqrt[n]{|a_n|} = (n^n)^{\frac{1}{n}} = n \to \infty \qquad$ so $R = 0$

and the series is convergent only for $z = 1$.

6. the function $\quad f(z) = \dfrac{1}{z^2 + iz}$ is analytic for $z \neq 0, -i \qquad$ around $z_0 = -i$

$$f(z) = \frac{1}{z^2+iz} = \frac{1}{z+i} \cdot \frac{1}{z} = \frac{1}{z+i} \cdot \frac{1}{(z+i)-i}$$

for a) we need positive powers of $(z+i)$ for $\dfrac{1}{(z+i)-i}$:

$$f(z) = \frac{1}{z+i} \cdot \frac{1}{(z+i)-i} = \frac{1}{z+i} \cdot \frac{1}{-i} \cdot \frac{1}{1 - \frac{(z+i)}{i}} = \frac{i}{z+i} \sum_{n=0}^{\infty} \frac{1}{i^n} (z+i)^n$$

using $\displaystyle\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ for $|w| < 1$,where $w = \dfrac{z+i}{i} \qquad$ separate $n = 0$

$$f(z) = \frac{i}{z+i} + \sum_{n=1}^{\infty} \frac{1}{i^{n-1}} (z+i)^{n-1} = \frac{i}{z+i} + \sum_{k=0}^{\infty} (-i)^k (z+i)^k \text{ for } 0 < |z+i| < 1$$

thus $b_1 = i \qquad a_9 = (-i)^9 = -i(-1)^4 = -i$

for b) we need negative powers only:

$$f(z) = \frac{1}{z+i} \cdot \frac{1}{(z+i)-i} = \frac{1}{(z+i)^2} \cdot \frac{1}{1 - \frac{i}{z+i}} = \frac{1}{(z+i)^2} \sum_{n=0}^{\infty} \frac{i^n}{(z+i)^n}$$

as above with $w = \dfrac{i}{z+i} \qquad f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{i^n}{(z+i)^{n+2}} = \sum_{k=2}^{\infty} \frac{-i^k}{(z+i)^k}$ for $1 < |z+i|$

Note: we can also use Partial Fractions as below.

7. For the centre $z_0 = 2i$ we need Partial Fractions:

$$f(z) = \frac{1}{z^2 + iz} = \left(\frac{1}{z} - \frac{1}{z+i}\right)\frac{1}{i} = \frac{-i}{z} + \frac{i}{z+i} = \frac{-i}{(z-2i)+2i} + \frac{i}{(z-2i)+3i}$$

first fraction in negative powers , second in positive powers of $(z - 2i)$

$$f(z) = \frac{-i}{(z-2i)} \cdot \frac{1}{1+\frac{2i}{z-2i}} + \frac{i}{3i} \cdot \frac{1}{1+\frac{z-2i}{3i}} = \frac{-i}{(z-2i)} \sum_{n=0}^{\infty} \frac{(-1)^n (2i)^n}{(z-2i)^n} + \frac{i}{3i} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3i)^n} (z-2i)^n$$

using $\sum_{n=0}^{\infty} (-1)^n w^n = \frac{1}{1+w}$ for $|w| < 1$ so the first one is valid for $\left|\frac{2i}{z-2i}\right| < 1$

the second one for $\left|\frac{z-2i}{3i}\right| < 1$ together $2 < |z - 2i| < 3$

finally

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^n i^{n+1}}{(z-2i)^{n+1}} + \sum_{n=0}^{\infty} \frac{\left(\frac{-1}{i}\right)^n}{3^{n+1}} (z-2i)^n = \sum_{k=1}^{\infty} \frac{(-1)^k 2^{k-1} i^k}{(z-2i)^k} + \sum_{n=0}^{\infty} \frac{i^n}{3^{n+1}} (z-2i)^n$$

thus $b_k = (-i)^k 2^{k-1}, k = \pm 1, \pm 2, ...$ $\quad a_n = \frac{i^n}{3^{n+1}}$ $\quad n = 0, 1, 2, ...$