PMAT 421 WINTER 08

Midterm 1 90 minutes SOLUTION

1. Express $\dfrac{\left(-1+i\sqrt{3}\right)^4}{(-1-i)^6}$ in the form $a+ib$,where a, b,real .

   polar form: $-1+i\sqrt{3} = 2e^{i\frac{2}{3}\pi}$ $\left(-1+i\sqrt{3}\right)^4 = 2^4 e^{i\frac{8}{3}\pi} = 2^4 e^{i2\pi} e^{i\frac{2}{3}\pi} = 2^4 e^{i\frac{2\pi}{3}}$

   also $-1-i = \sqrt{2}e^{-i\frac{3}{4}\pi}$ $(-1-i)^6 = \left(\sqrt{2}\right)^6 e^{-i\frac{3}{4}\pi\cdot 6} = 2^3 e^{-i\frac{9}{2}\pi} = 2^3 e^{-i4\pi} e^{-i\frac{\pi}{2}} =$ $2^3 e^{-i\frac{\pi}{2}}$

   together

   $$\frac{\left(-1+i\sqrt{3}\right)^4}{(-1-i)^6} = 2e^{i\pi\left(\frac{2}{3}+\frac{1}{2}\right)} = 2e^{i\pi\frac{7}{6}} = 2e^{i\pi}e^{i\frac{\pi}{6}} = -2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = -\sqrt{3} - i.$$

2. Describe (sketch) the set $S = \{z; \operatorname{Im}\frac{1}{z} \geq \frac{1}{2}\}$.

   Is it open,closed,connected? Explain.Find the set of all accumulation points of S.

   for $z \neq 0$ $\operatorname{Im}\frac{1}{z} = -\dfrac{y}{x^2+y^2} \geq \dfrac{1}{2}$ $-2y \geq x^2+y^2$ $1 \geq x^2+(y+1)^2$

   circular disk with radius 1 , center at $(0,-1)$ without the origin

   so the set is bounded , connected, neither open nor closed since the boundary is the circle incl 0

   the set of accumulations points is the whole circular disk incl 0.

3. Find all z satisfying $z^3 = -8$ n the form $a+ib$,where a, b,real .

   using polar form $-8 = 8e^{i\pi(1+2k)}$ $z = 2e^{i\frac{\pi}{3}(1+2k)}$

   $k=0$ $z_1 = 2e^{i\frac{\pi}{3}} = 1+i\sqrt{3}$ $k=1$ $z_2 = 2e^{i\pi} = -2$

   and $k=-1$ $z_3 = 2e^{-i\frac{\pi}{3}} = 1-i\sqrt{3}.$

4. Evaluate the following limit if it eists:

   (a) $\displaystyle\lim_{z\to i}\frac{z^6+1}{i+z^3} = \left(\text{"}\frac{0}{0}\text{"}\ \frac{\text{analytic}}{\text{analytic}} \to L'H.R.\right) = \lim_{z\to i}\frac{6z^5}{3z^2} = 2i^3 = -2i$

   (b) $\displaystyle\lim_{z\to 0}\frac{\operatorname{Re} z}{\bar{z}}$ DNE

   since for $z = x \neq 0$ $\dfrac{\operatorname{Re} z}{\bar{z}} = \dfrac{x}{x} = 1$

   and for $z = iy \neq 0$ $\dfrac{\operatorname{Re} z}{\bar{z}} = 0$

   .

5. Solve for all z : $\qquad e^{iz} = 2i.$

since $e^{iz} = e^{-y}e^{ix} = 2e^{i\pi(\frac{1}{2}+2k)} \qquad e^{-y} = 2$ and $x = \frac{\pi}{2} + 2k\pi$

together $\qquad z = \frac{\pi}{2} + 2k\pi - i\ln 2$ for any integer k

OR using log $\qquad iz = \log(2i) = \ln 2 + i\left(\frac{\pi}{2} + 2k\pi\right) \qquad z = -i\ln 2 + \frac{\pi}{2} + 2k\pi.$

6. For $f(z) = z^2 \operatorname{Im} z$ find all z for which f is differentiable ,then find $f'(z)$.

Where is f analytic? Explain.

for $z \neq 0 \qquad \operatorname{Im} z = \dfrac{f(z)}{z^2}$ so if f is diff. and we know that z^2 is diff then $\operatorname{Im} z$ will be also diff.

but we know it is NOT true therefore f is NOT diff for $z \neq 0$

for $z = 0 \qquad f'(0) = \lim_{z\to 0}\dfrac{f(z)-f(0)}{z} = \lim_{z\to 0}\dfrac{z^2 \operatorname{Im} z}{z} = \lim_{z\to 0} z \operatorname{Im} z = 0$

OR **use C.R. conditions**

$f(z) = (x+iy)^2\, y \qquad u(x,y) = x^2y - y^3 \qquad v(x,y) = 2xy^2$ and

$u_x = 2xy = v_y = 4xy \to xy = 0$

$u_y = x^2 - 3y^2 = -v_x = -2y^2 \to x^2 = y^2 \to x = \pm y$

together only $\qquad (0,0)$

all partials are continuus everywhere so at $z = 0$ the function is diff

and $f'(0) = u_x(0,0) + iv_x(0,0) = 0$

f is NOWHERE anlytic since it is diff. only at one point and we need an open set.

7. For the function $\qquad f(z) = \dfrac{z-i}{z} \qquad$ find the domain D,then $f'(z)$ in D;

then find the real function u, v such that $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$.

Finally, verify that $\qquad f'(z) = u_x(x,y) + iv_x(x,y)$ in D.

obviously $\qquad D = \{z \neq 0\} \qquad f(z) = 1 - \dfrac{i}{z}$ and by Rules $f'(z) = \dfrac{i}{z^2}$

also $\qquad f(z) = 1 - \dfrac{i\bar{z}}{|z|^2} = 1 - \dfrac{ix+y}{x^2+y^2}$

thus $\qquad u(x,y) = 1 - \dfrac{y}{x^2+y^2} \qquad v(x,y) = -\dfrac{x}{x^2+y^2}$

$$u_x = \frac{2xy}{(x^2+y^2)^2} \qquad v_x = -\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{(x^2-y^2)}{(x^2+y^2)^2}$$

finally

$$f'(z) = \frac{i}{z^2} = \frac{i(\bar{z})^2}{|z|^4} = \frac{i(x^2-y^2-2ixy)}{(x^2+y^2)^2} = \frac{2xy + i(x^2-y^2)}{(x^2+y^2)^2} = u_x + iv_x.$$

8. Show that if u and v are harmonic in D and v is a harmonic conjugate of u in D then uv is harmonic in D.

given:

$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0 \qquad u_x = v_y \qquad u_y = -v_x \qquad$ in D

by Product Rule

$(uv)_{xx} = (u_x v + u v_x)_x = u_{xx} v + 2u_x v_x + u v_{xx} \qquad (uv)_{yy} = (u_y v + u v_y)_y = u_{yy} v + 2u_y v_y + u v_{yy}$

togeher

$(uv)_{xx} + (uv)_{yy} = (u_{xx} + u_{yy})\, v + 2\,(u_x v_x + u_y v_y) + u\,(v_{xx} + v_{yy}) = 0 + 2\,(-u_x u_y + u_y u_x) + 0 = 0$