Pmat 421
Assignment # 1

1. $\left(\dfrac{2+i}{3+4i}+\dfrac{2i}{3-4i}\right)^2=\left(\dfrac{(2+i)(3-4i)+2i(3+4i)}{(3+4i)(3-4i)}\right)^2=$

$=\left(\dfrac{6-8i+3i+4+6i-8}{3^2+4^2}\right)^2=$

$=\left(\dfrac{2+i}{25}\right)^2=\frac{1}{625}(2+4i-1)=\frac{3}{625}+i\frac{4}{625}.$

2. For $z=x+iy, w=a+ib,\qquad z\bar{w}=(x+iy)(a-ib)=(ax+by)+i(ay-bx)$

and $\qquad \bar{z}w=(x-iy)(a+ib)=(ax+by)+i(-ay+bx)$

they are equal if real parts are the same and imaginery parts as well

$(ay-bx)=-(ay-bx)$ thus $(ay-bx)=0$

it means both real or if $by\neq 0$

or $\frac{a}{b}=\frac{x}{y}=r \qquad z=y(r+i) \qquad w=b(r+i)$

i.e. $\qquad w=sz$,where s is real,$s=\frac{b}{y}$.

3. $|z+i|\leq|z+2|\rightarrow x^2+(y+1)^2\leq(x+2)^2+y^2$

$2y+1\leq 4x+4 \qquad y\leq 2x+\frac{3}{2}$ the set is below and on the line,

the boundary is the line included, so the set is closed and simply connected

4. $\{(\operatorname{Re} z)^2>1\}=\{|x|>1\}=\{x>1\}\cup\{x<-1\}$

the boundary is two vertical lines $x=\pm 1$,accm.pts $S'=\{(\operatorname{Re} z)^2\geq 1\}$

the set is open and not connected

5. $-1+i=\sqrt{2}e^{i\frac{3}{4}\pi} \qquad (-1+i)^8=\left(\sqrt{2}\right)^8 e^{i\frac{3}{4}\pi\cdot 8}=2^4e^{i6\pi}=2^4=16$

$1-i\sqrt{3}=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=2e^{-i\frac{\pi}{3}} \qquad \left(1-i\sqrt{3}\right)^5=2^5e^{-i\frac{\pi}{3}\cdot 5}=2^5e^{i\frac{\pi}{3}(1-6)}=2^5e^{i\frac{\pi}{3}}e^{-i2\pi}=2^5e^{i\frac{\pi}{3}}$

:$(-1+i)^8\left(1-i\sqrt{3}\right)^5=2^4 2^5 e^{i\frac{\pi}{3}}=2^9\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2^8+2^8\sqrt{3}i$

check:$(-1+i)^8=\left((-1+i)^2\right)^4=(2i)^4=2^4(-1)^2=2^4$

$\left(1-i\sqrt{3}\right)^5=\left(1-i\sqrt{3}\right)\left[\left(1-i\sqrt{3}\right)^2\right]^2=\left(1-i\sqrt{3}\right)\left[\left(-2-2i\sqrt{3}\right)\right]^2=$

$=\left(1-i\sqrt{3}\right)4\left(1+i\sqrt{3}\right)^2=4\left(1-i\sqrt{3}\right)\left(-2+2i\sqrt{3}\right)=-8\left(1-i\sqrt{3}\right)^2=2^4\left(1+i\sqrt{3}\right)$

6. $\left(\frac{1+i}{1-i}\right)^3 - 2i = \left(\frac{(1+i)^2}{1+1}\right)^3 - 2i = i^3 - 2i = -3i$ so $Arg = -\frac{\pi}{2}$,generally

$\left(\frac{1+i}{1-i}\right)^3 - 2i = 3e^{i\pi\left(\frac{1}{2}+2k\right)}$, k is any integer,

then $\sqrt{\left(\frac{1+i}{1-i}\right)^3 - 2i} = \sqrt{3}e^{i\pi\left(-\frac{1}{4}+k\right)} = \pm\sqrt{3}e^{-i\pi\frac{1}{4}} = \pm\sqrt{\frac{3}{2}}\,(1-i)$

7. For $z = |z|\,e^{i\theta}$ where $\theta = Arg(z)$ must be in$(-\pi,\pi]$ $Arg\,(\bar{z}) = -Arg\,(z)$.

$\bar{z} = |z|\,e^{-i\theta}$ so $-\theta$ is not Arg only if $\theta = \pi$,the formula is valid for all z except $z = x \leq 0$;

similarily if both $Arg(z), Arg(w) \in \left(\frac{\pi}{2},\pi\right)$ then $Arg(z)+Arg(w) \in (\pi, 2\pi)$ not Arg

generally it must $Arg(z) + Arg(w) = Arg(zw) + 2k\pi$ for some value of k.

similarily for $\left(-\pi, -\frac{\pi}{2}\right)$.

8. we have to prove two implications:if $\arg z = \arg w$ then $|z+w| = |z| + |w|$

and if $|z+w| = |z| + |w|$ then $\arg z = \arg w$ which is equivalent to

if $\arg z \neq \arg w$ then $|z+w| \neq |z| + |w|$

Case I: if $\arg z = \arg w$ then $z = rw$ for some $r > 0$ and $|z+w| = (r+1)\,|w| = |z| + |w|$

Case II: if $\arg z \neq \arg w$ and $z = -rw, r > 0$ then $|z+w| = |1-r|\,|w| \neq (1+r)\,|w|$

Case III:if $\arg z \neq \arg w$ and $z \neq \pm rw$ then we can draw a triangle with sides being vectors z and w,

and the third side being $z + w$ (vector addtion) ,clearly $|z+w| \neq |z| + |w|$

OR

$|z+w| = |z| + |w|$ iff $|z+w|^2 = (|z| + |w|)^2$ iff $\operatorname{Re}(z\bar{w}) = |z\bar{w}|$ since

L.S. $(z+w)(\bar{z}+\bar{w}) = |z|^2 + z\bar{w} + \bar{z}w + |w|^2 = |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 =$

$$= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2$$

R.S. $(|z| + |w|)^2 = |z|^2 + 2\,|z|\,|\bar{w}| + |w|^2 = |z|^2 + 2\,|z\bar{w}| + |w|^2$

and $\operatorname{Re}(z\bar{w}) = |z\bar{w}|$ iff $\operatorname{Re}(z\bar{w}) > 0$ and $\operatorname{Im}(z\bar{w}) = 0$ iff $\arg(z\bar{w}) = 0$

and $\arg(z\bar{w}) = \arg(z) + \arg(\bar{w}) = \arg(z) - \arg(w) = 0$

9. Solving $z^5 = -1$;express $-1 = e^{i\pi(1+2k)}$ then $(-1)^{\frac{1}{5}} = e^{i\frac{\pi}{5}(1+2k)}$

we need 5 different values of k

$k=0 \qquad z_1=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$

$k=2 \qquad z_2=e^{i\pi}=-1$

$k=1 \qquad z_3=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$

$k=-1 \qquad z_4=e^{-i\frac{\pi}{5}}=\bar{z}_1=\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}$

$k=-2 \qquad z_5=e^{-3i\frac{\pi}{5}}=\bar{z}_3=\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}$

the angle between roots is 72 degrees or $\frac{2}{5}\pi$ and we have a symmetry -reflection in the x -axis

10. De Moivre's Theorem: $\qquad R.S. \qquad e^{i\theta 5}=\cos(5\theta)+i\sin(5\theta)$

$$L.S.\ \left(e^{i\theta}\right)^5=(\cos\theta+i\sin\theta)^5=\sum_{k=0}^{k=5}\left(\begin{array}{c}5\\k\end{array}\right)\cos^{5-k}\theta\cdot i^k\sin^k\theta$$

we need to compare the imaginary parts i.e. only odd $k=1,3,5$.

$$i^3=-i \qquad i^5=i \qquad \left(\begin{array}{c}5\\3\end{array}\right)=10, \left(\begin{array}{c}5\\1\end{array}\right)=5, \left(\begin{array}{c}5\\5\end{array}\right)=1$$

$$\sin(5\theta)=5\cos^4\theta\cdot\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$$