PMAT 421
Practice Midterm 2-Solution

For 1)

Evaluate $\int\limits_c \bar{z}\ dz$ where the curve c is a part of $|z-i|=2$ from $3i$ to $-2+i$.

Since $\bar{z}$ is nowhere analytic we have to used the definition of the integral first,parametrization of the circle : $z(t)=i+2e^{it}\quad t\in\left[\frac{\pi}{2},\pi\right]$

then $dz=z'(t)\ dt=2ie^{it}\ dt$ and $\oint\limits_c \bar{z}\ dz=2i\int\limits_{\frac{\pi}{2}}^{\pi}\left(-i+2e^{-it}\right)e^{it}\ dt=$

$=2\left[\frac{e^{it}}{i}\right]_{\frac{\pi}{2}}^{\pi}+4i\left[\pi-\frac{\pi}{2}\right]=-2i\left[e^{i\pi}-e^{i\frac{\pi}{2}}\right]+2i\pi=-2i\left[-1-i\right]+2i\pi=$

$=-2+2i\left(\pi+1\right).$

For 2)

(a)

$$\int\limits_c \frac{e^z}{z^2+4}dz=\int\limits_c \frac{e^z}{z+2i}\cdot\frac{1}{z-2i}dz=2\pi i\left(\frac{e^z}{z+2i}\right)_{z=2i}=\frac{\pi}{2}e^{2i}=\frac{\pi}{2}\left(\cos 2+i\sin 2\right)$$

since only $z_0=2i$,a simple pole, is inside $|z-i|=2$ use CIF- Cauchy Integral formula

(b)

since both singular points are in we have to
split $|z|=3$ into two small circles around $z=\pm 2i\qquad c_{1,2}:|z\mp 2i|=\frac{1}{2}$
then as above

$$\int\limits_c \frac{e^z}{z^2+4}dz=\int\limits_{c_1}\frac{e^z}{z^2+4}dz+\int\limits_{c_2}\frac{e^z}{z^2+4}dz=\int\limits_{c_1}\frac{e^z}{z+2i}\cdot\frac{1}{z-2i}dz+\int\limits_{c_2}\frac{e^z}{z-2i}\cdot\frac{1}{z+2i}dz=$$

$$=\frac{\pi}{2}e^{2i}+2\pi i\left(\frac{e^z}{z-2i}\right)_{z=-2i}=\frac{\pi}{2}e^{2i}-\frac{\pi}{2}e^{-2i}=\pi i\sin 2.$$

For 3)

For $\int\limits_c \frac{2z+1}{z^2-z}\ dz$

in a) $\qquad c$ is the line segment from i to 2 not passing through $z=0,1$

$\frac{2z+1}{z^2-z}=\frac{3}{z-1}-\frac{1}{z}$ thus $F(z)=3Log(z-1)-Logz$

in the plane minus the branch cut $\{\text{Re}(z-1)\leq 0,\text{Im}\,z=0\}$

so the line segment from i to 2 is in the domain and

$$\int\limits_i^2 \frac{2z+1}{z^2-z}\ dz=F(2)-F(i)=-\frac{5}{2}\ln 2-\frac{7}{4}\pi i$$

since $F(2)=3\ln 1-\ln 2=-\ln 2$

$F(i)=3Log(i-1)-Log(i)=3\left[\ln\sqrt{2}+\frac{3}{4}\pi i\right]-\ln 1-\frac{\pi}{2}i=\frac{3}{2}\ln 2+\frac{7}{4}\pi i;$

in b) $\qquad c$ is the circle $|z|=\frac{1}{2}$ with only $z=0$ inside

$$\int\limits_c \frac{2z+1}{z^2-z}\ dz=\int\limits_c \frac{2z+1}{z-1}\cdot\frac{1}{z}\ dz=2\pi i\left[\frac{2z+1}{z-1}\right]_{z=0}=-2\pi i.$$

For 4) For $\oint\limits_{|z|=2} \frac{\cos z}{z^3}\, dz$

the integrand function is analytic for $z \neq 0$
so we can use the extended Cauchy Integral formula $n = 2$:
$\oint\limits_{|z|=2} \frac{\cos z}{z^3}\, dz = 2\pi i \cdot \frac{1}{2!} g''(0) = -\pi i$
where $g(z) = \cos z$, $g\prime(z) = -\sin z$, $g''(z) = -\cos z$

For 5)
For $\int\limits_c \frac{Log\ z}{z} dz$
in (a)
the line segment from 1 to $1-i$ does not cross the principal branch cut
we can find $F(z)$ in the domain "complex plane minus cut"

$F(z) = \frac{1}{2}\left(Logz\right)^2$ then $\int\limits_1^{1-i} \frac{Log\ z}{z} dz = F(1-i) - F(1) = \frac{1}{2}\left[Log^2(1-i) - Log^2 1\right] =$

$= \frac{1}{2}\left[\ln\sqrt{2} - i\frac{\pi}{4}\right]^2 = \frac{1}{8}\left(\ln 2\right)^2 - \frac{\pi^2}{32} - i\frac{\pi}{8}\ln 2$ by Fund Th.of Calculus
in (b)
the curve $|z| = 1$ crosses the branch cut so by Definition:
$z(t) = e^{it}$ $t \in [-\pi, \pi]$ $z'(t) = ie^{it}$ and
$\int\limits_c \frac{Log\ z}{z} dz = \int\limits_{-\pi}^{\pi} \frac{Loge^{it}}{e^{it}} ie^{it} dt = i\int\limits_{-\pi}^{\pi} (it) dt = 0$ (odd function)

For 6)
Evaluate
$\int\limits_c z\,|z-1|\ dz$ where the curve c is a part of $|z-1| = 5$ where $\operatorname{Im} z \geq 0$.
Since $z\,|z-1|$ is nowhere analytic we have to used the definition of the integral
first,parametrization of the curve : $z(t) = 1 + 5e^{it}$ $t \in [0, \pi]$
then $dz = z'(t)\ dt = 5ie^{it}\ dt$ and $\int\limits_c z\,|z-1|\ dz = 5\int\limits_0^{\pi} \left(1 + 5e^{it}\right) 5ie^{it}\ dt =$

$= 25\left[e^{it}\right]_0^{\pi} + 125\left[\frac{e^{2it}}{2}\right]_0^{\pi} = 25\left[-1-1\right] + 0 = -50.$

For 7)
We know that $\left|\int\limits_c \frac{e^{iz}}{\bar{z}} dz\right| \leq ML$

in a) $L = \frac{1}{2} 2R\pi = 4\pi$ and $M = \max\left|\frac{e^{iz}}{\bar{z}}\right| = \max\frac{e^{-y}}{|z|} = \frac{e^4}{4}$
since $y \in [-4, 4]$ together the estimate is $ML = \pi e^4$;
in b)
$L = \sqrt{3^2 + 4^2} = 5$ for M maximize top, minimize bottom:
$\max e^{-y} = e^0 = 1$ since $y \in [0, 4]$

$\min|z| = d-$distance from the origin to the line segment
draw the right angle triangle with sides $3,4,5$,vertices $-3,0,4i$
the area of the triangle is $A=12/2=6$ and also $A=\frac{5\cdot d}{2}\to d=\frac{12}{5}$
together $\dfrac{e^{-y}}{|z|}\le\dfrac{5}{12}$ and the estimate is $ML=\dfrac{25}{12}$.
Note:
It could be shown that $\max\dfrac{e^{-y}}{\sqrt{x^2+y^2}}=\dfrac{1}{3}$ on the line $3y-4x=12$
by investigating min of the reciprocal squared:
$h(y)=e^{2y}\left(y^2+(\frac{3}{4}y-3)^2\right)$ on $[0,4]$,but it is harder.

For 8)

$\left|\int_c e^{\bar z\operatorname{Im}z}dz\right|\le ML=2\pi Re^{\frac{R^2}{2}}$ since $L=2\pi R$ and to find M :
$\left|e^{\bar z\operatorname{Im}z}\right|=e^{xy}$ since $\operatorname{Re}(\bar z\operatorname{Im}z)=\operatorname{Re}(xy-iy^2)=xy$
we need maximum of $f(x,y)=xy$ on the constraint $g(x,y)=x^2+y^2=R^2$
solve $\nabla f=\lambda\nabla g$ $\quad y=\lambda 2x$ $\quad x=\lambda 2y$ $\quad \frac{y}{x}=\frac{x}{y}$ $\quad y=\pm x$
(minima on the line $y=-x$)
maxima on the line $y=x\to x^2+y^2=R^2\to 2x^2=R^2$

so at the points $\left(\pm\dfrac{R}{\sqrt2},\pm\dfrac{R}{\sqrt2}\right)$ max.value is $\dfrac{R^2}{2}$ and then $M=e^{\frac{R^2}{2}}$

For 9)

For $\int_c z^{-\frac12}\,dz$ where $z^{-\frac12}$ is the branch where $\arg z\in[0,2\pi)$

we can use Fund.Theorem $\quad F(z)=2z^{\frac12}$
for any z **not** on the branch cut $\{z;\operatorname{Im}z=0,\operatorname{Re}z\ge0\}$!!!
thus $\quad \int_c z^{-\frac12}\,dz=F(i)-F(-1-i)=2e^{\frac12\log i}-2e^{\frac12\log(-1-i)}=$
(evaluate $\log(-1-i)=\ln\sqrt2+i\frac{5\pi}{4}$ and $\log i=\ln 1+i\frac{\pi}{2}$)
$=2\left[e^{i\frac{\pi}{4}}-e^{\ln\sqrt[4]{2}+i\frac58\pi}\right]=2\left[\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}-\sqrt[4]{2}\left(\cos\frac58\pi+i\sin\frac58\pi\right)\right]=$
$=\sqrt2-2^{\frac54}\cos\frac{5\pi}{8}+i\left(\sqrt2-2^{\frac54}\sin\frac58\pi\right)$
also $e^{i\frac58\pi}=e^{i\frac12\pi}\,e^{i\frac18\pi}=i(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8})=-\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}$

For 10)

$\sum_{n=1}^{\infty}n!\left(\dfrac{z}{n}\right)^n=\sum_{n=1}^{\infty}n!\dfrac{1}{n^n}z^n$
use Ratio Test $\left|\dfrac{a_{n+1}}{a_n}\right|=\left|\dfrac{(n+1)!}{(n+1)^{n+1}}\cdot\dfrac{n^n}{n!}\right|=(\dfrac{n}{n+1})^n=\left(1-\dfrac{1}{n+1}\right)^n\to\dfrac{1}{e}=L$

using the limit $\left(1+\frac{a}{n}\right)^n\to e^a$ as $n\to+\infty$
we know that $R=\frac{1}{L}=e$ $\quad$ it means the series is abs.convergent for $|z|<e$
and is abs. divergent for $|z|>e$.

For 11)

the function $f(z)=\dfrac{1}{z^2+z}$ is analytic except at $z=0$ and $z=-1$.

if the center is -1 we have two possible domains
$0 < |z+1| < 1$ and $|z+1| > 1$
since the distance from -1 to the nearest sing point 0 is 1
Now ,the point $3i$ is in the latter one ,we need negative powers of $(z+1)$

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z+1} \cdot \frac{1}{z+1-1} = \frac{1}{(z+1)^2} \cdot \frac{1}{1-\frac{1}{z+1}} = \frac{1}{(z+1)^2} \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^n =$$

(using the sum of geom. series $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ for $|w| < 1, w = \frac{1}{z+1}$)

$$= \sum_{n=0}^{\infty} \frac{1}{(z+1)^{n+2}} = \sum_{k=2}^{\infty} \frac{1}{(z+1)^k} \text{ for } \left|\tfrac{1}{z+1}\right| < 1 \text{ i.e. } |z+1| > 1.$$

For 12)

Defined for $z \neq \pm i$ denote $w = \frac{1}{z^2+1}$
then the series is a geometric one ,convergent for $|w| < 1$ and
the sum is $\sum_{n=1}^{\infty} w^n = \frac{w}{1-w}$ thus the original series is convergent for $|z^2+1| > 1$

and the sum $\sum_{n=1}^{\infty} \left(\frac{1}{z^2+1}\right)^n = \dfrac{\frac{1}{z^2+1}.}{1-\frac{1}{z^2+1}} = \frac{1}{z^2}$ for $z \neq 0, \pm i$

So far we have 3 holes $z = 0, \pm i$
Now the condition $|z^2+1| > 1$ is satisfied if both $|z \pm i| > 1$
since $|z^2+1| = |z-i|\ |z+i|$
therefore any point outside the circles $|z \pm i| = 1$ will be OK
in more detail:
$|z^2+1| > 1$ means $\quad (x^2-y^2+1)^2 + 4x^2y^2 > 1$
so e.g.
if $x = 0 \quad |1-y^2| > 1$ no $y \in (-1,1)$ since $1-y^2 > 1$ never true
for $|y| > 1 \quad y^2-1 > 1$ gives $|y| > \sqrt{2}$ on the y-axis;
if $y = 0 \quad x^2+1 > 1$ any x;the whole x- axis except the origin
generally if we solve $\quad (x^2-y^2+1)^2 + 4x^2y^2 > 1$
$x^4+y^4-2x^2y^2+2x^2-2y^2+1+4x^2y^2 > 1$
$(x^2+y^2)^2 + 2(x^2-y^2) > 0$ the curve $(x^2+y^2)^2 + 2(x^2-y^2) = 0$
in polar coordinates $\quad r^2 = -2\cos 2\theta$.....lemniscata=the set of all points
with the product of distances to two foci $\pm i$ being constant
therefore all points outside that lemniscata satisfied the condition;
this set is a little bigger than outside two circles.

For 13)

the function $f(z) = \frac{z}{z^2+z-2}$ is analytic except at $z = 1, -2$
for a) the centre is 1 so in powers of $(z-1)$
in $0 < |z-1| < 3$ —distance from 1 to -2

$$f(z) = \frac{z}{z^2+z-2} = \frac{z}{(z-1)(z+2)} = \frac{1}{(z-1)} \cdot \frac{(z+2)-2}{(z+2)} =$$

$$= \frac{1}{(z-1)}\left[1 - \frac{2}{(z+2)}\right] = \frac{1}{(z-1)}\left[1 - \frac{2}{3+(z-1)}\right] =$$

$$= \frac{1}{(z-1)}\left[1-\frac{2}{3}\cdot\frac{1}{1+\frac{z-1}{3}}\right] = \frac{1}{(z-1)} - \frac{2}{3}\frac{1}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n} =$$

(separate $n=0$)

$$= \frac{1}{(z-1)} - \frac{2}{3}\frac{1}{z-1} - \frac{2}{3}\sum_{n=1}^{\infty}(-1)^n\frac{(z-1)^{n-1}}{3^n} = \frac{1}{3\,(z-1)} + \frac{2}{9}\sum_{k=0}^{\infty}(-1)^k\frac{(z-1)^k}{3^k}$$

OR P.Fr. first

$$\frac{z}{z^2+z-2} = \frac{A}{z-1} + \frac{B}{z+2} =$$

$$= \frac{1}{3\,(z-1)} + \frac{2}{3(z+2)} = \frac{1}{3\,(z-1)} + \frac{2}{3}\cdot\frac{1}{(z-1)+3} =$$

$$= \frac{1}{3\,(z-1)} + \frac{2}{9\left(1+\frac{z-1}{3}\right)} = \frac{1}{3\,(z-1)} + \frac{2}{9}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}$$

using $\displaystyle\frac{1}{1+w} = \sum_{n=0}^{\infty}(-1)^n w^n$ for any $|w|<1$ where $w=\frac{z-1}{3}$

thus $b_{-1} = \frac{1}{3}$ and $a_n = \frac{2(-1)^n}{3^{n+2}}, a_9 = \dfrac{-2}{3^{11}}$.

for b)

the centre is 0 so in powers of z; we need P.Fr.

$$f(z) = \frac{z}{z^2+z-2} = \frac{1}{3\,(z-1)} + \frac{2}{3(z+2)} = \frac{1}{3z\left(1-\frac{1}{z}\right)} + \frac{2}{3\cdot 2\left(1+\frac{z}{2}\right)} =$$

$$= \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} = \sum_{k=1}^{\infty}\frac{1}{3z^k} + \sum_{n=0}^{\infty}\frac{(-1)^n}{3\cdot 2^n}z^n$$

the first series is covergent for $\left|\frac{1}{z}\right| < 1$ and the second one for $\left|\frac{z}{2}\right| < 1$

together $1 < |z| < 2$ as required;

$b_k = \frac{1}{3}$ for $k = 1, 2, ..; a_n = \dfrac{(-1)^n}{3\cdot 2^n}$ for $n = 0, 1, 2, ..$