Pmat 421
Parctice Midterm 1-Solution

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1. For $f(z) = \dfrac{z}{3\bar{z}-2} = \dfrac{x+iy}{3x-2-3iy} = \dfrac{(x+iy)(3x-2+3iy)}{(3x-2)^2+9y^2}$

   so $u(x,y) = \dfrac{3x^2-2x-3y^2}{(3x-2)^2+9y^2}$ and $v(x,y) = \dfrac{6xy-2y}{(3x-2)^2+9y^2}$

   in the domain $D = \{z \neq \frac{2}{3}\}$

   it is not onto C since $w = \frac{1}{3}$ is missing from the range

   we know that the real function $y = \dfrac{x}{3x-2}$ has a horizontal asymp $y = \frac{1}{3}$

   therefore $\dfrac{x}{3x-2} \neq \dfrac{1}{3}$ check a complex solution

   solve $\frac{1}{3} = \dfrac{z}{3\bar{z}-2}$ for z :

   $u = \frac{1}{3}$ $v = 0 \to 2y(3x-1) = 0$

   Case I: $y = 0$ thus $\dfrac{z}{3\bar{z}-2} = \dfrac{x}{3x-2}$ as above

   Case II: $x = \frac{1}{3}$ and

   $1 = 3u = \dfrac{-1-9y^2}{1+9y^2} \to 1+9y^2 = -1-9y^2$ impos.

2. $e^{iz}+3 = 0 \to e^{iz} = -3 \to e^{-y}e^{ix} = 3e^{i\pi}$

   so $e^{-y} = 3$ and $x = \pi + 2k\pi$

   thus $z = \pi(2k+1) - i\ln 3$

   OR apply log

   $iz = \log(-3) = \ln 3 + i\pi + i2k\pi$ $z = -i\ln 3 + \pi + 2k\pi, k$ is any integer

3. take two points $z_1, z_2 \in S = \{|z| < \pi\}$ and assume that $e^{z_1} = e^{z_2}$

   then $|e^{z_1}| = |e^{z_2}| \to \operatorname{Re} z_1 = \operatorname{Re} z_2$ since real exp. f . is one-to-one

   thus $e^{i \operatorname{Im} z_1} = e^{i \operatorname{Im} z_2} \to \cos(\operatorname{Im} z_1) = \cos(\operatorname{Im} z_2), \sin(\operatorname{Im} z_1) = \sin(\operatorname{Im} z_2)$

   from the properties of real trigs only possible if $\operatorname{Im} z_1 = \operatorname{Im} z_2 + 2k\pi$

   but $|\operatorname{Im} z_1|, |\operatorname{Im} z_2| < \pi$ so $k = 0$ and $z_1 = z_2$

   Not one-to-one on $\bar{S} = \{|z| \leq \pi\}$ since $e^{i\pi} = e^{-i\pi}$.

4. For $z \neq 0$ the branch of $\arg z \in [0, 2\pi)$

$$\arg z = \begin{cases} \arctan \frac{y}{x} & \text{for } x > 0, y \geq 0 \\ \frac{\pi}{2} & \text{for } x = 0, y > 0 \\ \arctan \frac{y}{x} + 2\pi & \text{for } x > 0, y \leq 0 \\ \frac{3}{2}\pi & \text{for } x = 0, y < 0 \\ \arctan \frac{y}{x} + \pi & \text{for } x < 0, y \text{ any} \end{cases} .$$

5. Prove that $|z| \geq \frac{1}{\sqrt{2}} (|\text{Re}\, z| + |\text{Im}\, z|)$.

   square both sides: $2\left(x^2 + y^2\right) \geq |x|^2 + 2|xy| + |y|^2$, $x^2 + y^2 - 2|xy| = \left(|x|^2 - |y|^2\right)^2 \geq 0$

   which is always true,then we can go back,using $x^2 = |x|^2$ for real numbers.

6. Sketch the set $S = \{0\} \cup \left\{ z\ ;\ \left| \frac{\text{Re}\, z}{\text{Im}\, z} \right| \geq 1 \right\}$; find the boundary ∂S.

   first , $\text{Im}\, z \neq 0$ so x-axis is NOT in the set, BUT the origin is;

   then in the first quadrant : $x > 0, y > 0$ we have $x \geq y$

   above the x-axis and below and on the line $y = x$,including the origin

   now we have symmetry $x \leftrightarrow -x, y \leftrightarrow -y$

   we can see that boundary $\partial S = \{y = \pm x \text{ and } x - axis\}$,part is in ,part is out

   so the set S is **neither open nor closed**,also is **unbounded** and it is **connected** since

   we can go through the origin,also **simply connected** since the complement is connected through ∞.

7. Solve $(e^z + 1)^2 = e^z$

   $(e^z)^2 + e^z + 1 = 0$ so first solve the quadratic equation $w^2 + w + 1 = 0$ where $w = e^z$

   $w = \frac{1}{2}\left[-1 \pm i\sqrt{3}\right] = e^{i\left(\pm \frac{2}{3}\pi\right)}$

   then $\text{m}e^z = e^x e^{iy} = e^{i\left(\pm \frac{2}{3}\pi\right)} \rightarrow x = 0, y = \pm \frac{2}{3}\pi + 2k\pi$

   OR

   $z = \log w = \ln 1 + i \arg w = i(\pm \frac{2}{3}\pi + 2k\pi)$ for any integer k.

8. Find $\lim\limits_{z \to i} \frac{[\text{Im}(z - i)]^2}{z - i} = 0.$

   we can try $x = 0, y \to 1$, then $y = 1, x \to 0$ and limit seems to be 0

   let's try to prove it: $\left| \frac{[\text{Im}(z - i)]^2}{z - i} - 0 \right| = \left| \frac{[y - 1]^2}{x + i(y - 1)} \right| \leq |y - 1| \to 0$

since $\quad |x+i(y-1)| = \sqrt{x^2+(y-1)^2} \geq \sqrt{(y-1)^2} = |y-1|$

OR

$|\text{Im}\, w| \leq |w| \qquad \left|\dfrac{[\text{Im}(z-i)]^2}{z-i}\right| \leq \dfrac{|z-i|^2}{|z-i|} = |z-i| \to 0$

Note: we cannot use L'H.R.

9. Compare two functions $f(z) = |z|^2$ (complex valued of complex variable) and $g(x) = |x|^2$ (real valued of real variable).
   (a) Where is g continous ,and where is differentiable?
   (b) Where is f continuous and where is differentiable?
   (c) Where is f analytic?

   For a) $\quad g(x) = x^2$ and $g'(x) = 2x$ so g is continous and differentiable for any real x

   For b) $\quad f(z) = z\overline{z}$ a product of 2 cont. functions so f is cont. for any complex z

   also $f = u + iv$ where $u(x,y) = x^2 + y^2$ and $v = 0 \qquad$ both cont.functions

   partials $\quad u_x = 2x, u_y = 2y, v_x = v_y = 0$ all cont. functions

   Cauchy-Riemann cond. are satisfied **only** at $(0,0)$ so f is diff, at (0,0) and $f'(0) = 0$

   For c) $\quad f$ is nowhere analytic.

10. Show that $u(x,y) = x^2 - y^2 + x - y$ is harmonic everywhere , find a harmonic conjugate $v(x,y)$ and then $f(z) = u + iv$ in terms of z.

    partials $\quad u_x = 2x+1, u_{xx} = 2, u_y = -2y-1, u_{yy} = -2$ so $\triangle u = 0$ and u is harmonic everywhere.For $v : v_y = 2x+1$ and $v_x = 2y+1$

    so $\quad v = \int u_x dy = 2xy + y + c(x)$ then $v_x = 2y + c'(x) = 2y+1$

    thus $c'(x) = 1$ and $c(x) = x$,together $v = 2xy + y + x$

    therefore $f(z) = x^2 - y^2 + x - y + i\,(2xy + y + x) = z^2 + z + iz$

    ALSO

    we could have guessed that $u = \text{Re}\left(z^2 + z + iz\right)$ and then $v = \text{Im}\left(z^2 + z + iz\right)$

11. for a)

    $w = (3+2i)\,z + 2 - i = (3+2i)\,(z-i) + i\,(3+2i) + 2 - i = (3+2i)\,(z-i) + 2i$

    so $\quad w - 2i = (3+2i)\,(z-i)$ and $|w-2i| = |(3+2i)\,(z-i)| \leq \sqrt{26}$

    for b)

$w = (3+2i)\,z + 2 - i = w = (3+2i)\,(x+iy) + 2 - i$ so

$u = 2 + 3x - 2y$ and $v = 2x + 3y - 1$

now if $y = x \qquad u = 2 + x \qquad v = 5x - 1 \qquad u - 2 = \dfrac{v+1}{5}$

the image is the line $\qquad 5u - 11 = v$

also find the images of two points on the line y=x $(0,0) \to (2,-1)$

and $(1,1) \to (3,4)$ then a line passing through them has $m = 5$

and $\dfrac{v+1}{u-2} = 5 \qquad v + 1 = 5u - 10$ as above.

12. $\lim\limits_{z\to\infty} \dfrac{[\operatorname{Im} z]}{z}$ exists iff $\lim\limits_{z\to 0} \dfrac{\left[\operatorname{Im} \frac{1}{z}\right]}{\frac{1}{z}} = \lim\limits_{(x,y)\to(0,0)} (x+iy)\, \dfrac{-y}{x^2+y^2} DNE$

since $\lim\limits_{(x,y)\to(0,0)} \dfrac{-xy}{x^2+y^2} DNE \qquad x = 0, y \neq 0 \qquad u = 0$

and for $x = y \neq 0 \qquad u = -\frac{1}{2}$