PMAT 421 WINTER 07
FINAL 3 hours

Solution

Note: Give the answers ,if possible in the form $a+ib, a, b$ real.
Each question is for 10 marks.

1. Find all values of
(a) $\dfrac{1}{\left(-\sqrt{3}-i\right)^i} = e^{-i\log\left(-\sqrt{3}-i\right)} = e^{-i\left[\ln\left|-\sqrt{3}-i\right|+i\theta+i2k\pi\right]} = e^{-i\ln 2}e^{\frac{7}{6}\pi+2k\pi} =$
$= e^{\frac{7}{6}\pi+2k\pi}\left[\cos(\ln 2) - i\sin(\ln 2)\right]$ (also $\theta = -\frac{5}{6}\pi$);
(b) $arg\left[\dfrac{(1-i)^{10}}{(1+i)^6}\right] = arg\left[\dfrac{(1-i)^{2\cdot 5}}{(1+i)^{2\cdot 3}}\right] = arg\left[\dfrac{(-2i)^5}{(2i)^3}\right] = arg(4) = 2k\pi$
where k is any integer OR
$arg\left[\dfrac{(1-i)^{10}}{(1+i)^6}\right] = arg\,(1-i)^{10} - \arg(1+i)^6 + 2n\pi = 10\cdot\dfrac{-\pi}{4} - 6\cdot\dfrac{\pi}{4} + 2n\pi =$
$= -\dfrac{8}{2}\pi + 2n\pi = 2k\pi$

2. $\cos z = 3i \rightarrow \dfrac{e^{iz}+e^{-iz}}{2} = 3i \rightarrow w + \dfrac{1}{w} = 6i$ where $w = e^{iz}$
$w^2 - 6iw + 1 = 0 \qquad w_1 = \frac{6i+\sqrt{-40}}{2} = i\left(3+\sqrt{10}\right)$
and $w_2 = i\left(3-\sqrt{10}\right)$
$iz = \log i\left(3+\sqrt{10}\right) = \ln\left(3+\sqrt{10}\right) + i\frac{\pi}{2} + 2ki\pi$
$z = -i\ln\left(3+\sqrt{10}\right) + \frac{\pi}{2} + 2k\pi$ also
$iz = \log i\left(3-\sqrt{10}\right) = \ln\left(\sqrt{10}-3\right) - i\frac{\pi}{2} + 2li\pi$
$z = -i\ln\left(\sqrt{10}-3\right) - \frac{\pi}{2} + 2l\pi$, where k, l are any integers
also $z = -i\ln\left(\sqrt{10}-3\right) - \frac{\pi}{2} + 2l\pi = z = i\ln\left(\sqrt{10}+3\right) - \frac{\pi}{2} + 2l\pi,$
so together $z = i\ln\left(3+\sqrt{10}\right) \pm \frac{\pi}{2} + 2k\pi = i\ln\left(3+\sqrt{10}\right) + \frac{\pi}{2} + k\pi;$
we can always solve $\cos z = w_0$ for any complex w_0 since
a quadratic equation $w^2 - 2iw_0 w + 1 = 0$
has always at least one complex root by Fund.Th. of Algebra,
and $w \neq 0$ so $z = \frac{1}{i}\log w$.

3. For $f(z) = x^2 - y^2 - 2xy + i(x^2 - 2xy)$,where $z = x + iy, x, y$ real

   $u(x, y) = x^2 - y^2 - 2xy \qquad v(x, y) = (x^2 - 2xy)$

   Necessary condition for f to be differentiable:

   $u_x = 2x - 2y = v_y = -2x \to 4x = 2y$

   $u_y = -2y - 2x = -v_x = -2x + 2y \to y = 0, x = 0$

   necessary conditions: If C.R.are NOT satisfied then f is not differentiable.

   Sufficient condition: since partial derivatives are cont. everywhere and

   C.R. conditions are satisfied at the origin f is differentiable at the origin only.

   But it is analytic nowhere since the origin is not an open set.

4. $u(x, y) = x^3 - 3xy^2 - 2y + 5$

   is harmonic since $u_{xx} = (3x^2 - 3y^2)_x = 6x$ and $u_{yy} = (-6xy - 2)_y = -6x$ thus $\triangle u = 0$

   For a harmonic conjugate v : $\qquad v_y = u_x = 3x^2 - 3y^2$

   then $v = \int u_x dy + c(x) = 3x^2 y - y^3 + c(x)$

   check $v_x = 6xy + c'(x) = -u_y = 6xy + 2 \to c(x) = 2x$

   together $v = 3x^2 y - y^3 + 2x + const.$

   For $f(z)$ $\qquad$ substitute $x = \frac{1}{2}(z + \bar{z}) \qquad y = \frac{1}{2i}(z - \bar{z})$

   also we can guess to get $u = \operatorname{Re} f(z)$ it must $f(z) = z^3 + 2iz + 5$

   since $z^3 + 2iz + 5 = (x + iy)^3 + 2i(x + iy) + 5 =$

   $= x^3 + 3ix^2 y - 3xy^2 - iy^3 + i2x - 2y + 5$ then $\qquad v = \operatorname{Im} f(z)$.

5. First the Laurent series for $\quad f(z) = \dfrac{1}{z - 1} = \dfrac{1}{(z - i) + i - 1}$ around $z_0 = i$

   the distance to the sing point $z = 1$ is $|i - 1| = \sqrt{2}$ so we have two possibilities

   $|z - i| < \sqrt{2} \qquad$ or $\qquad |z - i| > \sqrt{2}-$ this one contains 2 since $|i - 2| = \sqrt{5}$

   then $\dfrac{1}{z - 1} = \dfrac{1}{(z - i) + i - 1} = \dfrac{1}{(z - i)} \cdot \dfrac{1}{1 - \frac{1-i}{z-i}} =$

   $= \dfrac{1}{(z - i)} \sum_{n=0}^{\infty} \dfrac{(1 - i)^n}{(z - i)^n} = (k = n + 1) = \sum_{k=1}^{\infty} \dfrac{(1 - i)^{k-1}}{(z - i)^k}$, for $\left|\frac{1-i}{z-i}\right| < 1$

   now differentiate

   $\dfrac{-1}{(z - 1)^2} = \sum_{k=1}^{\infty} \dfrac{-k(1 - i)^{k-1}}{(z - i)^{k+1}} \qquad (k + 1 = n) \ \dfrac{1}{(z - 1)^2} = \sum_{n=2}^{\infty} \dfrac{(n - 1)(1 - i)^n}{(z - i)^n}$

   and $\qquad b_n = (n - 1)(1 - i)^n$ for $n \geq 2$.

6. Classify all singular points z_k of $f(z) = \dfrac{\cos z - 1}{z^2(e^z - 1)}$ and then find all $Res(z_k)$.

   first $z = 0 \qquad$ it is a simple pole since

$$Res(0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0}\frac{\cos z - 1}{z^2}(\text{"}\frac{0}{0}\text{"L.H.R.})\cdot \lim_{z\to 0}\frac{z}{e^z - 1} = -\frac{1}{2} \neq 0$$

next.solve: $\quad e^z - 1 = 0 \qquad z_k = \log 1 = 2k\pi i, k \neq 0$,simple poles

$$Res(z_k) = \lim_{z\to z_k}(z - z_k) f(z) = \lim_{z\to z_k}\frac{\cos z - 1}{z^2} \lim_{z\to z_k}\frac{z - z_k}{e^z - 1}(\text{ L.H.R.}) =$$

$$= \frac{\cos(2ki\pi) - 1}{-4k^2\pi^2}\frac{1}{e^{2ki\pi}} = \frac{1 - \cosh(2k\pi)}{4k^2\pi^2} \neq 0.$$

7. (a)

since $\quad f(z) = \dfrac{1}{z}$ is analytic for $z \neq 0$ and $F(z) = \log z$

the **branch** with $\arg \in [0, 2\pi)$

by **Fund.Th. of Calculus** for any curve from $-1 - i$ to $i\sqrt{2}$ not crossing the branch cut

$$\int_c \tfrac{1}{z}dz = \log\left(i\sqrt{2}\right) - \log(-1 - i) = \ln\sqrt{2} + i\tfrac{\pi}{2} - \left[\ln\sqrt{2} + i\tfrac{5}{4}\pi\right] = -\tfrac{3}{4}i\pi.$$

(b) by definition:

$$\int_c \frac{1}{z}\,dz = -\int_{\frac{\pi}{2}}^{\frac{5}{4}\pi}\frac{1}{\sqrt{2}e^{-it}}\cdot i\sqrt{2}e^{it}dt = \left[\tfrac{e^{2it}}{2}\right]_{\frac{5}{4}\pi}^{\frac{\pi}{2}} = \tfrac{1}{2}\left(e^{\pi i} - e^{\frac{5}{2}\pi i}\right) = \frac{-1 - i}{2}.$$

since for $-c \qquad z(t) = \sqrt{2}e^{it} \qquad t \in \left[\frac{\pi}{2}, \frac{5}{4}\pi\right]$

from $-1 - i$ to $i\sqrt{2}$ $\qquad$ lying in the left half of the complex plane.

8. By L.Th.if $\sin z$ is entire and bounded it must be constant- contradiction.

9. First $I = \displaystyle\int_0^\infty \frac{\cos 2x}{x^2(x^2+1)}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos 2x}{x^2(x^2+1)}dx = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty}\frac{e^{2ix}}{x^2(x^2+1)}dx =$

$$= \tfrac{1}{2}\,\mathrm{Re}\left[\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{2ix}}{x^2(x^2+1)}dx\right]$$

then $f(z) = \dfrac{e^{2iz}}{z^2(z^2+1)}$ is analytic except at $z = 0$ double pole

and $z = \pm i$ simple poles; the curve γ is dented around 0

since the degree of the bottom $\geq$2+degree of the top we know that

$\dfrac{1}{z^2(z^2+1)} \to 0$ as $z \to \infty \qquad$ thus

$I = \frac{1}{2}\,\mathrm{Re}\{2\pi i Res_i(f) + \pi i\ Res_0(f)\}$

now,

$$Res(i) = \left[(z - i) f(z)\right]_{z=i} = \left[\frac{e^{2iz}}{z^2(z+i)}\right]_{z=i} = \frac{e^{-2}}{-2i}$$

$$Res\,(0) = [z^2 f(z)]'_{z=0} = \left[\frac{e^{2iz}}{(z^2+1)}\right]'_{z=0} = \left[\frac{2ie^{2iz}}{(z^2+1)} - \frac{e^{2iz}\cdot 2z}{(z^2+1)^2}\right]_{z=0} = 2i$$

thus

$I = \frac{1}{2}\,\mathrm{Re}\{2\pi i Res\,(i) + \pi i\ Res(0)\} = \dfrac{-\pi}{2e^2} - \pi.$

10. For $w = e^{\pi z}$ $\quad w' = \pi e^{\pi z} \neq 0$ for all z so the mapping is conformal for all z;

but is it NOT one-to-one since $f(z + 2ki) = e^{\pi z} e^{2ki\pi} = e^{\pi z} = f(z)$....$2i-$periodic

the range is all $w \neq 0$ since we can solve for $z = \frac{1}{\pi}\log w$ only if $w \neq 0$

generally, $w = e^{\pi x + \pi y i}$ $\quad u = e^{\pi x}\cos \pi y \quad v = e^{\pi x}\sin \pi y$

the image $y = \frac{1}{2}, x$ any $\quad w = e^{\pi x + \frac{\pi}{2} i} \quad u = 0, v = e^{\pi x} > 0$, upper half of v-axis

the image $y = 1, x$ any $\quad w = e^{\pi x + \pi i} \quad v = 0, v = -e^{\pi x} < 0$, left half of u-axis

a vertical line between $y = y_0 \in \left(\frac{1}{2}, 1\right)$, x any,e.g. $y_0 = \frac{3}{4}$

so $\pi y_0 \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(\pi y_0) > 0, \cos(\pi y_0) < 0$

for our example .$u = \frac{-1}{\sqrt{2}} e^{\pi x}$ $\quad v = \frac{1}{\sqrt{2}} e^{\pi x}$so $u = -v, v > 0$

generally

$u = e^{\pi x}\cos \pi y_0, v = e^{\pi x}\sin \pi y_0 \quad u = mv, v > 0, m < 0$ a ray in the second quadrant

together the image of the set$\left\{z; \frac{1}{2} \leq \mathrm{Im}\, z \leq 1\right\}$ the second quadrant of the u, v-plane including axes , excluding the origin