PMAT 421 WINTER 00
FINAL
SOLUTIONS

For 1a)

$(-1)^{1-i} = e^{(1-i)\log(-1)} = e^{(1-i)[\ln 1 + i(\pi+2k\pi)]} = e^{i(\pi+2k\pi)} \cdot e^{\pi(2k+1)} = -e^{\pi(2k+1)}$

for any integer k.

For 1b)

$\sin(i-\pi) = \frac{1}{2i}\left(e^{i(i-\pi)} - e^{-i(i-\pi)}\right) = \frac{-i}{2}\left(e^{-1}e^{-i\pi} - e^{1}e^{i\pi}\right) = \frac{i}{2}\left(\frac{1}{e} - e\right).$

For 2)

for $\sin z = -i$ we can use the definition of $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = -i$

then $e^{iz} - e^{-iz} = 2$;for $w = e^{iz}$ we get $w - \frac{1}{w} = 2$

and $w^2 - 2w - 1 = 0$

$w_{1,2} = \frac{2\pm\sqrt{8}}{2} = 1 \pm \sqrt{2}$ or $w_1 = \sqrt{2} + 1 > 0, w_2 = 1 - \sqrt{2} < 0$

$z_k = \frac{1}{i}\log w_1 = -i\left[\ln\left(\sqrt{2}+1\right) + i\left(2k\pi\right)\right] = 2k\pi - i\ln\left(\sqrt{2}+1\right)$

AND

$z_l = \frac{1}{i}\log w_2 = -i\left[\ln\left(\sqrt{2}-1\right) + i\left(\pi + 2l\pi\right)\right]$

$= -i\ln\left(\sqrt{2}-1\right) + (2l+1)\pi = (2l+1)\pi + i\ln\left(\sqrt{2}+1\right)$

for any integers k, l.

For 3)

the property of log :$\log z = -\log\frac{1}{z} + i2n\pi$ for certain $n, z \neq 0$

a)

to get $n = 0$ we need both $Arg\ z, Arg\frac{1}{z} \in (-\pi, \pi]$ but

$Arg\ \frac{1}{z} = -\arg z \in [-\pi, \pi)$

so we have to **exclude points on the cut** $z = x \leq 0$ where $Arg\ z = \pi$

b)

it means that $\arg\ z \in [0, 2\pi)$ then $\arg\ \frac{1}{z} \in (-2\pi, 0]$

so **only** for points on the cut where $\arg\ z = \arg\frac{1}{z} = -\arg z = 0$

i.e.$z = x > 0$ the relation is valid.

For 4)

$f(z) = \dfrac{z}{z+4}$ is analytic for $z \neq -4$ distance from i to -4

is $|i+4| = \sqrt{17}$ thus around $z_0 = i$ we have two possible domains

$|z-i| < \sqrt{17}$ or $|z-i| > \sqrt{17}$,distance from i to 10 is $\sqrt{101}$

thus we need the latter to get 10 in,so negative powers of $(z-i)$:

first

$$\frac{1}{z+4} = \frac{1}{z-i+4+i} = \frac{1}{z-i}\cdot\frac{1}{1+\frac{4+i}{z-i}} = \left(\text{ for } \left|\frac{4+i}{z-i}\right| < 1\right)$$

$$= \frac{1}{z-i}\sum_{n=0}^{\infty}\frac{(-1)^n(4+i)^n}{(z-i)^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(4+i)^n}{(z-i)^{n+1}}$$

now

$$\frac{z}{z+4} = \frac{z+4-4}{z+4} = 1 - \frac{4}{z+4} =$$

$$= 1 + \sum_{n=1}^{\infty}\frac{4(-1)^n(4+i)^n}{(z-i)^{n-1}} = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k 4(4+i)^{k-1}}{(z-i)^k}$$

for $|z-i|>\sqrt{17}$ and $b_2=4\,(4+i)$
ALSO
$f(z)=\dfrac{z}{z+4}=1-\dfrac{4}{z+4}=1-\dfrac{4}{z-i+4+i}=...$

For 5)

Is $|\sin z|\le 1$ for all complex z? The function $\sin z$ is entire i.e. analytic everywhere if it is bounded then by Liouville's Th. $\sin z=const$.....contradiction
"If f is entire and bounded i.e.analytic and for some $M>0$ $|f(z)|\le M$ for **all** z; then f is constant ." Also $|\sin in|=\frac{1}{2}\left|e^{-n}-e^{n}\right|\to\infty$ as $n\to\infty$

For 6)

$\int\limits_c \dfrac{1}{\sqrt{z}}\,dz$ where c is the curve from $-i$ to $1+i$ lying in the right half of the plane
we can parametrize a line segment or use Fund. Th.$F(z)=2\sqrt{z}\ =2e^{\frac{1}{2}Logz}$
in the complex plane excluding the cut =negative x-axis since $F'(z)=\dfrac{1}{\sqrt{z}}$there

so $\int\limits_c \dfrac{1}{\sqrt{z}}\,dz=F(1+i)-F(-i)=2\left[(1+i)^{\frac{1}{2}}-(-i)^{\frac{1}{2}}\right]$

principal branch means that $Arg\,(1+i)=\frac{\pi}{4}$ and $Arg\,(-i)=-\frac{\pi}{2}$ so
$(1+i)^{\frac{1}{2}}=e^{\frac{1}{2}Log(1+i)}=e^{\frac{1}{2}\ln\sqrt{2}+i\frac{\pi}{8}}=\sqrt[4]{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$
and $(-i)^{\frac{1}{2}}=e^{\frac{1}{2}Log(-i)}=e^{-i\frac{\pi}{4}}=\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$
finally $\int\limits_c \dfrac{1}{\sqrt{z}}\,dz=2^{\frac{5}{4}}\cos\frac{\pi}{8}-\sqrt{2}+i(2^{\frac{5}{4}}\sin\frac{\pi}{8}+\sqrt{2}).$

For 7)

$f(z)=\frac{1}{z}e^{z+\frac{2}{z}}$ has an essential singularity at $z=0$
$f(z)=\frac{1}{z}e^{z}e^{\frac{2}{z}}=\frac{1}{z}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+..\right)\left(1+\frac{2}{z}+\frac{2^2}{2!z^2}+\frac{2^3}{3!z^3}\right)$
so the residue: $1+2+\frac{2^2}{(2!)^2}+\frac{2^3}{(3!)^2}+...=\sum\limits_{n=0}^{\infty}\dfrac{2^n}{(n!)^2}.$

For 8)

$$\int\limits_0^\infty \frac{\cos\frac{\pi}{4}x}{x^4-16}dx=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos\frac{\pi}{4}x}{x^4-16}dx=\frac{1}{2}\,\mathrm{Re}\int\limits_{-\infty}^{\infty}\frac{e^{ix\frac{\pi}{4}}}{x^4-16}dx$$

$f(z)=\dfrac{1}{z^4-16}=\dfrac{1}{(z^2-4)\,(z^2+4)}$ is analytic
except at $z=\pm2$......on the real axis and $z=\pm2i$,only $z=+2i$ inside the curve
the curve $c=[-R,-2-\varepsilon]\cup c_\epsilon^{-2}\cup[-2+\varepsilon,2-\varepsilon]\cup c_\epsilon^{2}\cup[2+\varepsilon,R]\cup\Gamma_R$
with two"dents" - semicircles around $z=\pm2$, and big semi-circle $R>>2$
$\oint\limits_c f(z)e^{iz\frac{\pi}{4}}dz=2\pi iRes_{2i}$,since $\lim\limits_{z\to\infty}f(z)=0$ as $R\to\infty$,
and

$$\int\limits_0^\infty \frac{\cos\frac{\pi}{4}x}{x^4-16}dx=\frac{1}{2}\,\mathrm{Re}\int\limits_{-\infty}^{\infty}\frac{e^{ix\frac{\pi}{4}}}{x^4-16}dx=\frac{1}{2}\,\mathrm{Re}\{2\pi iRes_{2i}+\pi i\,[Res_{-2}+Res_{2}]\}=$$
$$=-\frac{\pi}{32}(e^{-\frac{\pi}{2}}+1)$$

$$\mathrm{Res}_{2i}=\left[(z-2i)\,f(z)e^{iz\frac{\pi}{4}}\right]_{z=2i}=\left[\frac{e^{iz\frac{\pi}{4}}}{(z^2-4)\,(z+2i)}\right]_{z=2i}=\frac{e^{-\frac{\pi}{2}}}{-32i}$$

$$\text{Res}_2 = \left[(z-2)\,f(z)e^{iz\frac{\pi}{4}}\right]_{z=2} = \left[\frac{e^{iz\frac{\pi}{4}}}{(z^2+4)\,(z+2)}\right]_{z=2} = \frac{e^{i\frac{\pi}{2}}}{32} = \frac{i}{32} = \frac{-1}{i32}$$

$$\text{Res}_{-2} = \left[(z+2)\,f(z)e^{iz\frac{\pi}{4}}\right]_{z=-2} = \left[\frac{e^{iz\frac{\pi}{4}}}{(z^2+4)\,(z-2)}\right]_{z=-2} = \frac{e^{-i\frac{\pi}{2}}}{-32} = \frac{-i}{-32} = \frac{-1}{i32}$$

For 9)

$$\int_0^{2\pi} \frac{\sin 3\theta}{5-3\sin\theta}d\theta = \text{Im}\int_0^{2\pi} \frac{e^{i3\theta}}{5-3\sin\theta}d\theta = I$$

by subst. $z = e^{i\theta}, \dfrac{dz}{iz} = d\theta, \sin\theta = \dfrac{1}{2i}\left(z-\dfrac{1}{z}\right), ek^{i3\theta} = z^3$

$$\oint_{|z|=1} \frac{z^3}{5-3\cdot\frac{z^2-1}{2iz}}\frac{dz}{iz} = \oint_{|z|=1} \frac{z^3}{5iz-\frac{3}{2}\cdot(z^2-1)}dz = \oint_{|z|=1} \frac{2z^3dz}{10iz-3z^2+3} =$$

$$= \oint_{|z|=1} \frac{-2z^3dz}{3z^2-10iz-3} \qquad \text{find the poles} : 3z^2-10iz-3=0$$

$z_{1,2} = \frac{10i\pm\sqrt{-100+36}}{6} = \frac{10i\pm 8i}{6} = \frac{1}{3}i$ or $3i$ and $3z^2-10iz-3 = 3\left(z-\frac{i}{3}\right)(z-3i)$

only $z_1 = \frac{i}{3}$ is inside the unit circle so by Residue Th.

$$\oint_{|z|=1} \frac{-2z^3dz}{3z^2-10iz-3} = 2\pi i\ \text{Res}_{\frac{i}{3}} = 2i\pi\left[\left(z-\tfrac{i}{3}\right)f(z)\right]_{z=\frac{i}{3}} = 2\pi i\left[\frac{-2z^3}{3\,(z-3i)}\right]_{z=\frac{i}{3}} =$$

$$= 2\pi i\cdot\frac{-2\left(\frac{i}{3}\right)^3}{-8i} = \frac{\pi}{2}\cdot\frac{-i}{27} \text{ and } I = \text{Im}\oint_{|z|=1}\ldots = \frac{-\pi}{54}.$$

For 10a)

the mapping $w = f(z) = z - \frac{1}{z}$ is conformal if $z \neq \pm i, 0$ since

$f'(z) = 1 + \dfrac{1}{z^2} = \dfrac{z^2+1}{z^2} \neq 0 \qquad w = f(\pm i) = \pm 2i$ (critical points)

For b)

for $|z| = 2 \qquad z(t) = 2e^{it}$ so $w = 2e^{it} - \frac{1}{2}e^{-it} = \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta$

$\left(\frac{2}{3}u\right)^2 + \left(\frac{2}{5}v\right)^2 = 1 \qquad$ an ellipse with $\pm 2i$ inside

For c)

if $z = iy, y \neq 0$ then $w = iy - \frac{1}{iy} = i\frac{y^2+1}{y} \qquad u = 0, v \neq 0$

so y-axis minus origin onto a part of v-axis minus origin

to find out which part find the range of $f(t) = \dfrac{t^2+1}{t}$

or $\left|\dfrac{t^2+1}{t}\right| \geq 2 \qquad t^2+1-2|t| = (|t|-1)^2 \geq 0$

thus $|v| \geq 2 \qquad z = \pm i \to w = \pm 2i$

For d)

$|z| = 1$ passes through $\pm i$ and $z = e^{i\theta}$

$w = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ so $u = 0, v \in [-2, 2]$ .. vertical line segment

Notice that y-axis and unit circles go through the critical points $z = \pm i$ and are prependicular to each other but images are parallel.