**Pmat 421 W 11**

**Assignment # 1 due by Mo,Jan. 24,2011 Solution**

Each questions is worth 5 points.

1. Express $z$ in the form $a+ib$, with $a,b$ real

$$z=\left(\frac{3+i}{2-i}+\frac{2+3i}{3-2i}\right)^2=\left(\frac{3+i}{2-i}\cdot\frac{2+i}{2+i}+\frac{2+3i}{3-2i}\cdot\frac{3+2i}{3+2i}\right)^2=\left(\frac{5+5i}{5}+\frac{0+13i}{13}\right)^2$$

$=(1+2i)^2=-3+4i.$

2. Describe/sketch the following sets:

**a)** $\mathrm{Re}(z+i)=5\to x=5$ vertical line;

**b)** $\mathrm{Im}(z+i)=5\to y+1=5, y=4$ horizontal line;

**c)** $|z+i|=5\to x^2+(y+1)^2=25$ circle with the centre at $(0,-1)\,, R=5.$

3. $z^2=1+2i\to \qquad x^2-y^2=1 \qquad 2xy=2\to y=\dfrac{1}{x}$

$x^2-\dfrac{1}{x^2}=1 \qquad x^4-x^2-1=0 \qquad x^2=\dfrac{1+\sqrt5}{2}>0$

so $x=\pm\sqrt{\dfrac{1+\sqrt5}{2}} \qquad y=\pm\sqrt{\dfrac{2}{1+\sqrt5}}=\pm\sqrt{\dfrac{\sqrt5-1}{2}}$ thus

$z=\pm\sqrt{\dfrac{1+\sqrt5}{2}}\pm i\sqrt{\dfrac{\sqrt5-1}{2}}$

4. $z=(-1+i)^6(\sqrt3-i)^5$

$(-1+i)^6=(\sqrt2e^{i\frac34\pi})^6=2^3e^{i\frac92\pi}=2^3e^{i\left(4+\frac12\right)\pi}=2^3e^{i\frac12\pi}$

$(\sqrt3-i)^5=(2e^{-i\frac{\pi}{6}})^5=2^5e^{-i\frac56\pi}$ together

$z=2^3e^{i\frac12\pi}2^5e^{-i\frac56\pi}=2^8e^{i\pi\left(\frac12-\frac56\right)}=2^8e^{-i\frac{\pi}{3}}$ so $Arg z=-\dfrac{\pi}{3}$..

5. Express $\left|z^2\right|=\left|x^2-y^2+2ixy\right|=\sqrt{\left(x^2-y^2\right)^2+4x^2y^2}=$

$=\sqrt{x^4-2x^2y^2+y^4+4x^2y^2}=\sqrt{\left(x^2+y^2\right)^2}=x^2+y^2=|z|^2$

then for $z\neq0$

$$\left|\frac1z\right|=\left|\frac{x-iy}{x^2+y^2}\right|=\sqrt{\frac{x^2}{\left(x^2+y^2\right)^2}+\frac{y^2}{\left(x^2+y^2\right)^2}}=\sqrt{\frac{1}{x^2+y^2}}=\frac{1}{|z|}$$

by Math.induction for $n$ positive integer:

$\left|z^{n+1}\right|=|z^n\cdot z|=|z^n|\,|z|=|z|^n\,|z|=|z|^{n+1}$

using $|zw|=|z|\,|w|$ and then math.induction assumption;

for negative $n = -m, m > 0 \qquad z^n = \left(\frac{1}{z}\right)^m$ so we can use

the statement above for $w = \frac{1}{z}$ instead of $z$. and $m$ instead $n$

6. Since $-\pi < Argz \le \pi, z \ne 0 \qquad |\pi - Argz| = \pi - Argz < \frac{\pi}{4}$

   and then $\qquad \frac{3}{4}\pi < Argz \le \pi \qquad$ so the set the wedge between two rays $y = -x, y > 0$ excluded and $y = 0, x < 0$ included.

7. First for $\qquad -8 - i8\sqrt{3}$ find the absolute value and argument

   $\left|-8 - i8\sqrt{3}\right| = 16$ and $\theta = \arctan\sqrt{3} - \pi = \frac{-2}{3}\pi$ so

   $\left(-8 - i8\sqrt{3}\right)^{\frac{1}{4}} = \left(16e^{i\pi\left(\frac{-2}{3}+2k\right)}\right)^{\frac{1}{4}} = 2e^{i\pi\left(-\frac{1}{6}+\frac{k}{2}\right)}$

   $k = 0 \qquad z_0 = 2e^{-i\frac{\pi}{6}} = 2\cos\frac{\pi}{6} - i2\sin\frac{\pi}{6} = \sqrt{3} - i$

   $k = 1 \qquad z_1 = z_0 e^{i\frac{\pi}{2}} = z_0 \cdot i = 1 + i\sqrt{3}$

   $k = 2 \qquad z_2 = z_0 e^{i\pi} = -z_0 = -\sqrt{3} + i$

   $k = 3$ or $-1 \qquad z_3 = z_0 e^{i\frac{3\pi}{2}} = z_0 \cdot (-i) = -1 - i\sqrt{3}.$

8. De Moivre Theorem:$\left(e^{i\theta}\right)^5 = e^{i5\theta}$ so $\mathrm{Re}\left(e^{i\theta}\right)^5 = \mathrm{Re}\, e^{i5\theta}$

   $(\cos\theta + i\sin\theta)^5 = \sum_{k=0}^{5} \binom{5}{k} \cos^{5-k}\theta \cdot i^k \sin^k\theta \qquad$ for real part

   $k = 0, 2, 4 \qquad \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta = \cos(5\theta)$

9. the formula $2\sin x\sin y = \cos(x - y) - \cos(x + y)$

   L.S.left-hand side $\qquad = 2\,\mathrm{Im}\, e^{ix} \cdot \mathrm{Im}\, e^{iy}$

   R.S.right-hand side $= \mathrm{Re}\, e^{i(x-y)} - \mathrm{Re}\, e^{i(x+y)} = \mathrm{Re}\left(e^{ix} \cdot e^{-iy}\right) - \mathrm{Re}\left(e^{ix} \cdot e^{iy}\right)$

   using the properties $\qquad e^{i(x+y)} = e^{ix} \cdot e^{iy} \qquad \overline{e^{ix}} = e^{-ix}$

   $\mathrm{Re}(zw) = \mathrm{Re}\, z \cdot \mathrm{Re}\, w - \mathrm{Im}\, z \cdot \mathrm{Im}\, w \qquad \mathrm{Im}\,\overline{z} = -\mathrm{Im}\, z$ and $\mathrm{Re}\,\overline{z} = \mathrm{Re}\, z$

   $\mathrm{R.S.} = \mathrm{Re}\left(e^{ix}\right) \cdot \mathrm{Re}(e^{-iy}) - \mathrm{Im}\left(e^{ix}\right) \cdot \mathrm{Im}(e^{-iy}) - \mathrm{Re}\left(e^{ix}\right) \cdot \mathrm{Re}(e^{iy}) + \mathrm{Im}\left(e^{ix}\right) \cdot \mathrm{Im}(e^{iy}) =$

   $= \mathrm{Re}\left(e^{ix}\right) \cdot \mathrm{Re}(e^{iy}) + \mathrm{Im}\left(e^{ix}\right) \cdot \mathrm{Im}(e^{iy}) - \mathrm{Re}\left(e^{ix}\right) \cdot \mathrm{Re}(e^{iy}) + \mathrm{Im}\left(e^{ix}\right) \cdot \mathrm{Im}(e^{iy}) =$
   $2\,\mathrm{Im}\left(e^{ix}\right) \cdot \mathrm{Im}(e^{iy})$ .

10. the set $S = \{z; z\overline{z} \ge |z|\} \to |z|^2 \ge |z| \to z = 0$ or . $|z| \ge 1$

    consits of two parts the origin and outside or on the unit circle

    thus the set is **NOT connected, unbounded,**

    the boundary is $\partial S = \{0\} \cup \{|z| = 1\}$ is in so the set is **closed**

    $z = 0$ is isolated therefore only, **boundary** point of $S$