Pmat 421 W09
Assignment # 1 solution

1. $z = \dfrac{(1+i)(3-i)(-2-i)}{i(3+4i)(5-i)} = \dfrac{(4+2i)(-2-i)}{i(3+4i)(5-i)} = \dfrac{(-2)(2+i)^2}{i(3+4i)(5-i)} = \dfrac{(-2)(2+i)^2}{i(3+4i)(5-i)} =$

   $= \dfrac{(-2)(3+4i)}{(3+4i)(1+5i)} = \dfrac{(-2)(1-5i)}{(1+5^2)} = \dfrac{-1}{13} + i\dfrac{5}{13}$

2. Find the sum $\sum_{n=0}^{300} i^n = \sum_{k=0}^{150} i^{2k} + \sum_{k=0}^{149} i^{2k+1} = \sum_{k=0}^{150} (-1)^k + i \sum_{k=0}^{149} (-1)^k =$

   $= [(1-1) + ... + (1-1) + 1] + i[(1-1) + .. + (1-1)] = 1 + i0 = 1$

   OR using

   $$1 + z + z^2 + .. + z^n = \frac{1 - z^{n+1}}{1-z} \qquad \text{for any } z \neq 1$$

   $\sum_{n=0}^{300} i^n = \dfrac{1 - i^{301}}{1-i} = \dfrac{1 - i(-1)^{150}}{1-i} = 1$

3. first the polar froms of $-1 - i = \sqrt{2}e^{-i\frac{3\pi}{4}}$ and $-\sqrt{3} + i = 2e^{i\frac{5\pi}{6}}$

   then $z = \dfrac{(-1-i)^3}{\left(-\sqrt{3}+i\right)^2} = \dfrac{2^{\frac{3}{2}} e^{-i\frac{9\pi}{4}}}{2^2 e^{i\frac{5}{3}\pi}} = 2^{-\frac{1}{2}} e^{i\pi\left(-\frac{9}{4} - \frac{5}{3}\right)} = \dfrac{1}{\sqrt{2}} e^{i\pi \frac{-47}{12}} =$

   $\dfrac{1}{\sqrt{2}} e^{i\pi \frac{-48+1}{12}} =$

   $= \dfrac{1}{\sqrt{2}} e^{i\frac{\pi}{12}}$ since $e^{-i4\pi} = 1$ $\quad z = \dfrac{1}{\sqrt{2}} \cos \dfrac{\pi}{12} + i\dfrac{1}{\sqrt{2}} \sin \dfrac{\pi}{12}$

   thus(principal branch) $Arg\ z = \dfrac{\pi}{12}$.

   .

4. L.S. $|z+w|^2 + |z-w|^2 = (z+w)(\bar{z} + \overline{w}) + (z-w)(\bar{z} - \overline{w}) =$

   $= z\bar{z} + z\overline{w} + w\bar{z} + \overline{w}w + z\bar{z} - z\overline{w} - w\bar{z} + \overline{w}w =$

   $= 2|z|^2 + 2|w|^2$ R.S. for any complex z and w.

5. Solve $\quad z^{\frac{4}{3}} + 2i = 0 \qquad z^4 = (-2i)^3 = -8i^3 = 8i = 8e^{i\frac{\pi}{2}+2ki\pi}$

so $\quad z = \sqrt[4]{8}e^{i\frac{\pi}{8}+\frac{2ki\pi}{4}} = \sqrt[4]{8}e^{i\frac{\pi}{8}+\frac{ki\pi}{2}}$

for $\ k=0 \qquad z_1 = \sqrt[4]{8}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$;

for $k=1 \qquad z_2 = \sqrt[4]{8}e^{i\frac{\pi}{8}+\frac{i\pi}{2}} = iz_1 = \sqrt[4]{8}\left(-\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}\right)$

for $k=-1 \qquad z_3 = \sqrt[4]{8}e^{i\frac{\pi}{8}-\frac{i\pi}{2}} = -iz_1 = \sqrt[4]{8}\left(\sin\frac{\pi}{8} - i\cos\frac{\pi}{8}\right)$;

for $\ k=2 \qquad z_4 = z_2 = \sqrt[4]{8}e^{i\frac{\pi}{8}+i\pi} = -z_1 = \sqrt[4]{8}\left(-\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

6. For $z \neq 0 \qquad$ (a) $\qquad \operatorname{Im}\frac{1}{z} = \operatorname{Im}\frac{x-iy}{x^2+y^2} = -\frac{y}{x^2+y^2} = -y = -\operatorname{Im} z$

so one solution $\ y=0 \to z = x$ is real; for $y \neq 0 \qquad x^2+y^2 = 1 \to |z| = 1$

the x- axis without the origin and the unit circle

(b) $\qquad \operatorname{Re}\frac{1}{z} = \frac{x}{x^2+y^2} = -x = -\operatorname{Re} z \qquad$ only solution $x = 0$

thus $\qquad z = iy, y \neq 0 \qquad$ the y-axis without the origin

7. Find all roots $\ z^3 = -8 = 8e^{i(\pi+2k\pi)}$ so $z = \sqrt[3]{8}e^{i\pi\frac{1+2k}{3}}$

for $\ k=0 \qquad z_1 = \sqrt[3]{8}e^{i\frac{\pi}{3}} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + i\sqrt{3}$

for $\ k=1 \qquad z_2 = \sqrt[3]{8}e^{i\pi} = -2$

for $\ k=-1 \qquad z_3 = \sqrt[3]{8}e^{-i\frac{\pi}{3}} = 2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = 1 - i\sqrt{3}$

8. this particular branch is defined as follows:

$$\arg z = \begin{cases} \arctan\frac{y}{x} & for \quad x>0, y\geq 0 \\ \frac{\pi}{2} & for \quad x=0, y>0 \\ \arctan\frac{y}{x}+\pi & for \quad x<0, y \text{ any} \\ \frac{3}{2}\pi & for \quad x=0, y<0 \\ \arctan\frac{y}{x}+2\pi & for \quad x>0, y<0 \end{cases}$$

Investigate $\qquad \arg\bar{z} = -\arg z$

if $\qquad \arg z = \theta \in [0, 2\pi)$ then $-\theta \in (-2\pi, 0]$

so only if $\theta = 0$ we get back to the correct interval thus only if $z = x > 0$ $\arg \bar{z} = -\arg z = 0$ for other z $\arg \bar{z} = -\arg z + 2\pi$

9. we have NO accumulation points since the set $\{(-i)^n\,; n = 1, 2, 3, ...\}$ consists
   only of 4 isolated points $\pm 1 \pm i$
   for $n = 4k$ $(-i)^n = \left[(-i)^4\right]^k = 1$
   for $n = 4k + 1$ $(-i)^n = -i\left[(-i)^4\right]^k = -i$
   for $n = 4k + 2$ $(-i)^n = (-i)^2\left[(-i)^4\right]^k = -1$
   for $n = 4k + 3$ $(-i)^n = (-i)^3\left[(-i)^4\right]^k = -i.$

10. the set $\left\{z; \operatorname{Im}\frac{1}{z} \geq \frac{1}{6}\right\}$ for $z \neq 0$ $\frac{-y}{x^2 + y^2} \geq \frac{1}{6}$ $-6y \geq x^2 + y^2$ $9 \geq x^2 + (y+3)^2$
    circular disk with radius 3 , center $(0, -3)$ without $(0, 0)$ OR $\{z; z \neq 0, |z + 3i| \leq 3\}$
    the boundary is the whole circle $|z + 3i| = 3$ incl. $z = 0$
    not the whole boundary is in $(z = 0)$ so the set is NOT closed;
    some part of boundary is in so the set is NOT open;
    it is bounded and connected;the set of all accumulation points $\{z; |z + 3i| \leq 3\}$