PMAT 431, Fall 2004

Solution to Assignment 4

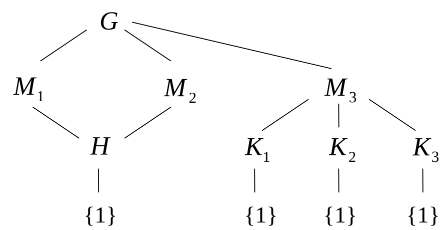
1. §8.4 # 14: If $|G| = p^2q$ where p and q are primes, show that G is not simple.

   Solution. If $p = q$ then $|G| = p^3$ so $\{1\} \neq Z(G) \lhd G$. Otherwise we have $n_q = 1,\ p,\ p^2$. If $n_q = 1$ we are done. If $n_q = p^2$ there are $p^2(q-1)$ elements of order q, leaving exactly p^2 elements. Hence $n_p = 1$ and we are done. So it suffices to show $n_q = p$ is impossible. But if $n_q = p$, then $|G : N(Q)| = p$ where Q is a Sylow q-subgroup. Hence there is a homomorphism $\theta : G \to S_p$. If $\ker\ \theta = \{1\}$ then $p^2q \,|\, p!$, whence $p|(p-1)!$, a contradiction.

2. §9.1 # 5: Find all composition series for $C_4 \times C_2$.

   Solution.

   Write $G = C_4 \times C_2$. If M is a maximal subgroup then $|M| = 4$ so M is cyclic or M is the Klein group. If $C_4 = \langle a \rangle$, $|a| = 4$, and $C_2 = \langle b \rangle$, $|b| = 2$, the elements of G of order 4 are $(a,1), (a,b), (a^3,1)$ and (a^3,b), so the cyclic maximal subgroups are $M_1 = \langle (a,1) \rangle = \langle (a^3,1) \rangle$ and $M_2 = \langle (a,b) \rangle = \langle (a^3,b) \rangle$. These have a unique subgroup $H = \langle (a^2,1) \rangle$ of order 2 leading to composition series $G \supset M_1 \supset H \supset \{1\}$ and $G \supset M_2 \supset H \supset \{1\}$. On the other hand, the only elements of order 2 in G are $(a^2,1), (a^2,b)$ and $(1,b)$, so $M_3 = \{(1,1), (a^2,1), (1,b), (a^2,b)\}$ is the unique maximal subgroup isomorphic to the Klein group. This has three subgroups of order 2: $K_1 = \langle (a^2,1) \rangle$, $K_2 = \langle (1,b) \rangle$ and $K_3 = \langle (a^2,b) \rangle$. This leads to three composition series $G \supset M_3 \supset K_1 \supset \{1\}$, G$\supset M_3 \supset K_2 \supset \{1\}$ and $G \supset M_3 \supset K_3 \supset \{1\}$. There are thus five composition series.

3. Section 9.2 # 9 and #10: Let p and q be primes in $\mathbb{Z}$.

   #9. Show that every group of order p^2q is solvable. [Hint: Problem 1 above.]

   #10. Show that every group of order p^2q^2 is solvable.

   Solution.

   #9. If $|G| = p^2q$ let $K \lhd G$, $K \neq \{1\}$, $K \neq G$ by Problem 1. Then $|K| = p,\ q,\ p^2$ or pq, so $|G/K| = pq,\ p^2,\ q$ or p. Thus both K and G/K are either abelian or of order pq, and hence are both solvable by Example 5. So G is solvable by Theorem 4.

   #10. If $|G| = p^2q^2$ let $K \lhd G$, $K \neq \{1\}$, $K \neq G$ by Example 9, Section 8.4. Then $|K| = p,\ q,\ p^2,\ q^2,\ pq,\ p^2q$ or pq^2, so $|G/K| = pq^2,\ p^2q,\ q^2,\ p^2,\ pq,\ q$ or p. Then both are (abelian or) solvable by Example 5 and the preceding exercise. So G is solvable by Theorem 4.

4. Section 6.2 # 7: Find the minimal polynomial of $u = \sqrt{3} - i$: (a) over $\mathbb{R}$; (b) $\mathbb{Q}$.

   Solution.

   (a). $(u - \sqrt{3})^2 = (-i)^2 = -1$ so $u^2 - 2\sqrt{3}u + 4 = 0$. We claim $m(x) = x^2 - 2\sqrt{3}x + 4 \in \mathbb{R}[x]$ is the minimal polynomial. Its roots in $\mathbb{C}$ are $\sqrt{3} \pm i$ and neither are in $\mathbb{R}$. So it is irreducible in $\mathbb{R}[x]$, as required.

   (b). $u^2 = 3 - 1 - 2\sqrt{3}i$ so $(u^2-2)^2 = -12$, $u^4 - 4u^2 + 16 = 0$. We claim $m(x) = x^4 - 4x^2 + 16$ is the minimal polynomial of u over $\mathbb{Q}$. It has no root in $\mathbb{Q}$; suppose $m(x) = (x^2 + ax + b)(x^2 + cx + d)$. Then $a + c = 0$, $b + d + ac = -4$, $ad + bc = 0$ and $bd = 16$. If $a = 0$ this gives $b^2 + 4b + 16 = 0$, a contradiction for $b \in \mathbb{Q}$. If $a \neq 0$ we get $b = d = \pm 4$ and $b + d - a^2 = -4$, again impossible. So $m(x)$ is irreducible.

5. Section 6.2 # 13(d): Find $[E : F]$ if $E = \mathbb{Q}(\sqrt[3]{3}, \sqrt{2})$ and $F = \mathbb{Q}(\sqrt{2})$.

   Solution. Put $u = \sqrt{3}$ so $E = F(u)$. We show that $[E : F] = 3$ by showing that $m(x) = x^3 - 3$ is irreducible in $F[x]$; that is $m(x)$ has no root in $F[x]$. The roots of $m(x)$ in $\mathbb{C}$ are u, uw and uw^2 where $w = e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3}i)$. Since w and w^2 are not in F, we show $u \notin F$ (then m has no root in F). But $u \in F$ means $3 = \deg_{\mathbb{Q}}(u)$ divides $\deg_{\mathbb{Q}}(\sqrt{2}) = 2$ by the Corollary to Theorem 5, a contradiction.