MIDTERM EXAMINATION WITH SOLUTIONS

PMAT 431 **FALL 2006**

1. (a) How many nonisomorphic abelian groups are there of order 21,600? [2 marks]

   (b) Let $R = \mathbb{Z}[\sqrt{-3}]$, and let $p = 1+\sqrt{-3}$. Show that p is irreducible in R, but not prime. [You may use the fact that the norm $N(a+b\sqrt{-3}) = a^2+3b^2$ satisfies $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{Z}[\sqrt{-3}]$.]

   Solution. (a). $21600 = 2^5 3^3 5^2$, so the types of the primary components are:

   $G(2)$: (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1) and (1,1,1,1,1)—7 in all

   $G(3)$: (3), (2,1), (1,1,1)—3 in all

   $G(5)$: (2), (1,1)—2 in all.

   So there are $7 \cdot 3 \cdot 2 = 42$ choices in all.

   (b). If $p = rs$ in R then $4 = N(p) = N(r)N(s)$. If $N(r) = 1$ then r is a unit; $N(r) = 2$ is impossible, and $N(r) = 4 \Rightarrow N(s) = 1 \Rightarrow s$ is a unit. Hence p is irreducible. Write $\bar{p} = 1 - \sqrt{-3}$ so that $p\bar{p} = 4 = 2 \cdot 2$. If p were prime then $p \mid 2$, say $p = 2q$ in R. Hence $4 = N(p) = N(2)N(q) = 4N(q)$, so q is a unit in R, whence $q = \pm 1$, whence $p = \pm 2$, a contradiction. [3 marks]

2. Let${}_R M$ be a module over an integral domain R, and write $T(M) = \{x \in M \mid ax = 0 \text{ for some } a \neq 0 \text{ in } R\}$. The module M is called *torsion* if $T(M) = M$, and M is called *torsion-free* if $T(M) = \{0\}$.

   (a) Show that $T(M)$ is a submodule of M. [2 marks]

   (b) Show that $M/T(M)$ is torsion-free. [2 marks]

   (c) If R is a PID and M is finitely generated, show that $M = T(M) \oplus K$ for some submodule K. [You may use the fact that finitely generated torsion-free modules over a PID are free.] [2 marks]

   Solution. Write $T(M) = T$ for convenience.

   (a). If $x, y \in T$ let $ax = 0 = by$ where $a \neq 0 \neq b$ in R. If $t \in R$ then $tx \in T$ because $a(tx) = 0$; and $x + y \in T$ because $ab(x + y) = 0$ and $ab \neq 0$ (as R is a domain).

   (b). If $x+T \in T(M/T)$ let $a(x+T) = 0$ with $a \neq 0$ in R. Hence $ax \in T$, say $c(ax) = 0$ for $0 \neq c \in R$. Thus $(ca)x = 0$, so $x \in T$ because $ca \neq 0$. Hence $x + T = 0$ in M/T.

   (c). The coset map $\phi : M \to M/T$ is an onto module homomorphism, so M/T is also finitely generated. Moreover, M/T is free by (b) and the hint. Hence M/T is projective (Theorem 7 §7.1) so the map ϕ "splits", that is $M = ker(\phi) \oplus K$ for some submodule K.

3. (a) Show that there are exactly two groups of order 325 up to isomorphism. [3 marks]

   (b) Let $G = S_4 \times S_5$, where S_n denotes the symmetric group on n letters. Find the composition length, and the composition factors (with multiplicities) of G. [3 marks]

   Solution. (a). Let G be a group of order $325 = 5^2 \cdot 13$. So $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 13$, that is $n_5 = 1$. Hence let $P \lhd G$ where $|P| = 5^2$. Similarly $n_{13} \equiv 1 \pmod{13}$ and $n_{13} \mid 5^2$, so $n_{13} = 1$ and we have $Q \lhd G$ with $|Q| = 13$. Then $P \cap Q = \{1\}$ and so $|PQ| = |P|\,|Q| = |G|$. Hence $PQ = G$ and we have $G \cong P \times Q$ by Theorem 7 §8.1. But P is abelian by Theorem 8 §8.2, so $P \cong C_{25}$ or $P \cong C_5 \times C_5$. Since $Q \cong C_{13}$, we have $G \cong C_{25} \times C_{13}$ or $G \cong C_5 \times C_5 \times C_{13}$. [Note that $C_5 \times C_{13} \cong C_{65}$, so $G \cong C_5 \times C_{65}$ in this second case.

   (b). Write $K = \{\varepsilon, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ and $H = \{\varepsilon, (1\ 2)(3\ 4)\}$. Then $S_4 \supset A_4 \supset K \supset H \supset \{\varepsilon\}$ is a composition series of length 4 for S_4 with factors C_2, C_3, C_2, and C_2; and $S_5 \supset A_5 \supset \{\varepsilon\}$ is a composition series of length 2 for S_5 with factors C_2 and A_5. Hence the length of $S_4 \times S_5$ is $4 + 2 = 6$, and the factors are C_2, C_3, C_2, C_2, C_2 and A_5.

4. (a) Let $p \in \mathbb{Z}$ be a prime, and let G be a group that has a finite p-group $P \neq \{1\}$ as a homomorphic image. Show that G has a normal subgroup $K \lhd G$ of index $|G : K| = p$. [3 marks]

   (b) If $C \subseteq G$ are groups, write $C \subseteq^{ch} G$ if $\sigma(C) = C$ for all automorphisms $\sigma : G \to G$ (we say C is *characteristic* in G). Assume that $K \lhd G$. If $C \subseteq^{ch} K$ prove that $C \lhd G$. Conclude that $Z(K) \lhd G$ where $Z(K)$ denotes the centre of the group K. [3 marks]

   Solution. (a). Let $\theta : G \to P$ be an onto group homomorphism with kernel $H \lhd G$. Hence $G/H \cong P$ by the isomorphism theorem and so G/H is a finite p-group. By Theorem 9 §8.2 it follows that G/H has a normal subgroup X of index p, and $X = K/H$ for a noromal subgroup $K \lhd G$ by the correspondence theorem. Finally $|G : K| = |G/H : K/H| = p$, as required.

   (b). Let $a \in G$, and define $\sigma_a : G \to G$ by $\sigma_\alpha(g) = aga^{-1}$ for all $g \in G$. Then σ_a is an (inner) automorphism of G so, as $C \subseteq^{ch} G$, we have $\sigma_a(C) = C$. Since $a \in G$ was arbitrary, this shows that $C \lhd G$. The last comment follows because the center of any group is a characteristic subgroup.

5. (a) Let $K \lhd G$ be groups with K finite, and let P be a Sylow subgroup of K. Show that $G = KN$ where $N = N_G(P)$ is the normalizer of P in G. [Hint: Any two Sylow subgroups (for the same prime) are conjugate.] [3 marks]

   (b) Let $K \subseteq H \subseteq G$ be groups where $K \lhd G$ and $H \lhd G$. Use the isomorphism theorem to prove that $H/K \lhd G/K$ and $\dfrac{G/K}{H/K} \cong G/H$. [3 marks]

   Solution. (a). Let $a \in G$. Since $P \subseteq K$, we have $aPa^{-1} \subseteq aKa^{-1} = K$ (as $K \lhd G$). Since $aPa^{-1} \cong P$, they are both Sylow subgroups of K for the same prime, and so (Sylow II) are conjugate in K. Hence there exists $k \in K$ such that $k(aPa^{-1})k^{-1} = P$, that is $(ka)P(ka)^{-1} = P$. This shows that $ka \in N_G(P) = N$, whence $a \in KN$. As $a \in G$ was arbitrary, this shows that $G = KN$.

   (b). Define $\alpha : G/K \to G/H$ by $\alpha(gK) = gH$ for all $g \in G$. This is well defined because $gK = g_1K$ implies $g_1^{-1}g \in K \subseteq H$, whence $gH = g_1H$. Then α is clearly an onto group homomorphism, and so $G/H \cong (G/K)/ker(\alpha)$. But $ker(\alpha) = \{gK \mid gH = H\} = \{gK \mid g \in H\} = H/K$.

6. (a) Let G be a finite p-group where p is a prime. If $H \neq G$ is a subgroup of G, show that $H \neq N(H)$ where $N(H)$ is the normalizer of H in G. [Hint: Let H act on $X = \{xH \mid x \in G\}$ by $h \bullet xH = hxH$.] [3 marks]

   (b) Let G be a nonabelian group of order $|G| = p^3$ where $p \in G$ is a prime. Show that $G' = Z(G)$. [You may use the fact that if $G/Z(G)$ is cyclic then G is abelian.] [3 marks]

   Solution. (a) The action in the hint is well defined because $xH = yH \Rightarrow y^{-1}x \in H \Rightarrow (hy)^{-1}(hx) = y^{-1}x \in H \Rightarrow hxH = hyH$. Hence $p \mid (|X| - |X_f|)$ by Theorem 4 §8.3. But $|X| = |G : H|$ so $p \mid |X|$. It follows that $p \mid |X_f|$, so $|X_f| \neq 1$. But $H \in X_f$ here because $X_f = \{xH \mid h \bullet xH = xH \text{ for all } h \in H\}$. Hence let $xH \in X_f$ where $xH \neq H$ (that is $x \notin H$). Hence $hxH = xH$ for all $h \in H$; that is $x^{-1}hx = H$ for all $h \in H$; that is $x^{-1}Hx = H$; that is $x \in N(H)$. Since $x \notin H$ this shows that $H \subset N(H)$.

   (b). Since G is a finite p-group, we know (Theorem 7 §8.2) that $Z = Z(G) \neq \{1\}$. Hence $|Z| = p$, p^2 or p^3. We have $|Z| \neq p^3$ because G is not abelian. If $|Z| = p^2$ then $G/Z \cong C_p$ so (by the hint) G is abelian, a contradiction. Hence $|Z| = p$. But then $|G/Z| = p^2$ so G/Z is abelian, whence $G' \subseteq Z$. Since $G' \neq \{1\}$ because G is not abelian, and since $|Z| = p$, it follows that $G' = Z$.