PMAT 431, Fall 2004

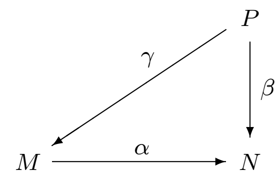
Solution to Assignment 2

1. §7.1 # 19: Show that the following conditions on a finitely generated module P are equivalent:

   Show that the following conditions on a finitely generated module P are equivalent:

   (1) P is projective.

   (2) P is isomorphic to a direct summand of a free module.
   (3) If α is onto in the diagram, then γ exists such that $\alpha\gamma = \beta$.
   (4) If $\alpha : M \to P$ is onto there exists γ: $P \to M$ such that $\alpha\gamma = 1_P$.

   Solution. (1)⇒(2). This is clear from Corollary 1 of Theorem 5.

   (2)⇒(3). By (2), we can identify P with a summand of a finitely generated free module F, say $F = P \oplus Q$, an internal direct sum. Define $\pi : F \to P$ by $\pi(p + q) = p$ for all $p \in P$ and $q \in Q$. Let $\{x_1, \cdots, x_n\}$ be a basis of F. Given the diagram we have $\beta\pi : F \to N$, so since α is onto, choose $m_i \in M$ such that $\alpha(m_i) = \beta\pi(x_i)$ for each i. By Theorem 5, there exists an R-homomorphism $\theta : F \to M$ such that $\theta(x_i) = m_i$ for each i. Then $\alpha\theta(x_i) = \alpha(m_i) = \beta\pi(x_i)$ for each i, so $\alpha\theta = \beta\pi$ because the x_i span F. But then, if $p \in P$, we obtain $\alpha\theta(p) = \beta\pi(p) = \beta(p)$ because $\pi(p) = p$. Hence the restriction $\gamma : P \to M$ given by $\gamma(p) = \theta(p)$ for $p \in P$, satisfies all our requirements.

   (3)⇒(4). In the diagram take $N = P$ and $\beta = 1_P$. Then (1) gives $\gamma : P \to M$ such that $\alpha\gamma = 1_P$.

   (4)⇒(1). Let $\alpha : W \to P$ be onto where W is free on a finite basis. By (4) let $\gamma : P \to W$ satisfy $\alpha\gamma = 1_P$. By Exercise 13 this means $W = \gamma(P) \oplus \ker(\alpha)$. Hence P is projective.

2. §7.2 # 10: Are the groups $Z_5 \oplus Z_{10} \oplus Z_{25} \oplus Z_{36} \oplus Z_{54}$ and $Z_{50} \oplus Z_{108} \oplus Z_{450}$ isomorphic?

   Solution.
   $$\begin{aligned} \mathbb{Z}_5 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{54} &= \mathbb{Z}_5 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{25} \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_9) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_{27}) \\ &= (\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_{27} \oplus \mathbb{Z}_9) \oplus (\mathbb{Z}_{25} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5). \\ \mathbb{Z}_{50} \oplus \mathbb{Z}_{108} \oplus \mathbb{Z}_{450} &= (\mathbb{Z}_2 \oplus \mathbb{Z}_{25}) \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_{27}) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{25}) \\ &= (\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_{27} \oplus \mathbb{Z}_9) \oplus (\mathbb{Z}_{25} \oplus \mathbb{Z}_{25}). \end{aligned}$$

   They have isomorphic 2-components and 3-components, but the 5-components have types $(2, 1, 1)$ and $(2, 2)$ respectively, and so are not isomorphic. Hence the groups are not isomorphic by Corollary 1 of Theorem 3.

3. §7.2 # 12: Let $K \subseteq M$ be modules. Show that

   (a) $T(K) = K \cap T(M)$.

   (b) If $K \subseteq T(M)$, show that $T(M/K) = T(M)/K$.

   Solution. (a) $T(K) = \{k \in K \mid |k| \neq 0\} = K \cap \{m \in M \mid |m| \neq 0\} = K \cap T(M)$.

   (b) If $m + K \in T(M/K)$, then $d(m + K) = 0$ for some $0 \neq d \in R$. Hence $dm \in K$ so, since $K \subseteq T(M)$, let $b(dm) = 0$ with $0 \neq b \in R$. Since $bd \neq 0$ (R is a domain), this shows that $m \in T(M)$, and hence that $m + K \in T(M)/K$. This proves that $T(M/K) \subseteq T(M)/K$. Conversely, if $m \in T(M)$ then $cm = 0$ with $0 \neq c \in R$. Hence $c(m + K) = 0$ in $T(M)/K$, so $m + K \in T(M/K)$. Thus $T(M)/K \subseteq T(M/K)$.

4. §7.2 # 18: If $\alpha : M \to N$ is a homomorphism of R-modules, show that $\alpha[T(M)] \subseteq T(N)$, and that there is a unique homomorphism $\bar{\alpha} : M/T(M) \to N/T(N)$ satisfying $\bar{\alpha}\varphi = \theta\alpha$ where $\varphi : M \to M/T(M)$ and $\theta : N \to T(N)$ are the coset maps.

   Solution. If $x \in T(M)$, say $dx = 0$ where $d \neq 0$, then $d \cdot \alpha(x) = \alpha(dx) = 0$ so $\alpha(x) \in T(N)$. Hence $\alpha[T(M)] \subseteq T(N)$. Now $\bar{\alpha}\varphi = \theta\alpha$ if and only if, for all $x + T(M)$ in $M/T(M)$,
   $$\bar{\alpha}[(x + T(M)] = \bar{\alpha}[\varphi(x)] = \theta[\alpha(x)] = \alpha(x) + T(N).$$
   Thus $\bar{\alpha}$ is unique if it exists. But $\bar{\alpha}$ is well defined by (*) because $\alpha[T(M)] \subseteq T(N)$, and it is clearly a homomorphism.

5. §8.1 # 9: If $K \lhd G$, show that the following conditions are equivalent.

   (1) The only subgroups H such that $K \subseteq H \subseteq G$ are $H = K$ and $H = G$.

   (2) G/K is cyclic and of prime order.

Solution. (1)⇒(2). By the correspondence theorem, (1) shows that $H = G/K$ has no subgroups except $\{1\}$ and H. If $0 \neq a \in H$, this means $H = \langle a \rangle$. Indeed: If $|a| = \infty$, then $\langle a^2 \rangle$ is a proper subgroup; so $|a| = n < \infty$. If $p \mid n$, then $\left|a^{n/p}\right| = p$ and $\langle a^{n/p} \rangle = H$.

(2)⇒(1). Given H as in (1), H/K is a subgroup of G/K. Then (2) implies $H/K = \{K\}$ or $H/K = G/K$. Thus $H = K$ or $H = G$ by the correspondence theorem.