PMAT 431, Fall 2004

Solution to Assignment 3

1. §8.1 # 19: If $H \lhd G$ and $K \lhd G$ show that $\frac{H}{H\cap K} \times \frac{K}{H\cap K} \cong \frac{HK}{H\cap K}$. [This extends Theorem 7.]

   Solution. Write $N = H \cap K$ for convenience. Then $N \lhd HK$ and we can define $\alpha : H \times K \to \frac{HK}{N}$ by $\alpha(h,k) = Nhk$. Proving that α is a homomorphism means showing $Nhk\,Nh_1k_1 = Nhh_1kk_1$ for all $h, h_1 \in H$, $k, k_1 \in K$. This is equivalent to $Nhkh_1k_1 = Nhh_1kk_1$, that is $hkh_1k_1(hh_1kk_1)^{-1} \in N$, that is $hkh_1k^{-1}h_1^{-1}h^{-1} \in N$. But $(kh_1k^{-1})h_1^{-1} \in Hh_1^{-1} = H$ and $k(h_1k^{-1}h_1^{-1}) \in kK = K$, so $h(kh_1k^{-1}h_1^{-1})h^{-1} \in hNh^{-1} = N$, as required. Now

   $$\ker\alpha = \{(h,k) \mid Nhk = N\} = \{(h,k) \mid hk \in N\}.$$

   Since $hk \in N \Rightarrow h \in Nk^{-1} \subseteq K$, we have $h \in H \cap K = N$, and similarly, $k \in H \cap K = N$. Hence $\ker\alpha = N \times N$. This shows $\frac{H\times K}{N\times N} \cong \frac{HK}{N}$. But $\frac{H}{N} \times \frac{K}{N} \cong \frac{H}{N} \times \frac{K}{N}$ by Example 7 §2.10.

2. §8.2 #9: If a finite group G has an element with exactly two conjugates, show that G is not simple.

   Solution. If $|\text{class } a| = 2$ then $|G : N(a)| = 2$ by Theorem 2. Hence $N(a)$ is normal in G. Since $N(a) \neq G$ we are done if $N(a) \neq \{1\}$. But $N(a) = \{1\}$ implies $a \in Z(G)$, so $Z(G) \neq \{1\}$ is normal in G. Since $Z(G) \neq G$, we are done. [G abelian implies class $a = G$ for all $a \in G$.]

3. §8.2 # 22: Let G be a finite group. If p is a prime, show that G has a normal subgroup of index p if and only if p divides $|G/G'|$, where G' is the commutator subgroup. [*Hint*: If p divides $|G/G'|$ apply Theorem 9 to the p-primary component of G/G'.]

   Solution. If $K \lhd G$ and $|G : K| = p$, then G/K is abelian so $G' \subseteq K$. Thus G/G' has a (normal) subgroup K/G' of index p, and so p divides $|G/G'|$. Conversely, write $H = G/G'$. If p divides $|H|$, it suffices to show that H has a normal subgroup of index p (it has the form K/G' so $K \lhd G$ and $|G : K| = p$ by the correspondence theorem). Now H is abelian so it has an element of order p by Cauchy's theorem. Thus the p-primary component $H(p) \neq \{1\}$. Since $H(p)$ is a p-group, let $M \lhd H(p)$, $|H(p) : M| = p$ by Theorem 9. But primary decomposition theorem shows (in multiplicative notation) that $H \cong H(p) \times N$ for some subgroup N of H. But $M \times N$ has index p in $H(p) \times N$ because $\frac{H(p)\times N}{M\times N} \cong \frac{H(p)}{M}$. This does it.

4. § 8.3 # 27: If G is a finite p-group, show that the number of nonnormal subgroups of G is a multiple of p.

   Solution. Let G act on $X = \{H \mid H \subseteq G \text{ is a subgroup}\}$ by conjugation: $a \cdot H = aHa^{-1}$. Then $X_f = \{H \mid a \cdot H = H \text{ for all } a\}$ is the set of normal subgroups of G. Hence the nonnormal subgroups are partitioned into nonsingleton conjugacy classes, and each of these has $|G : N(H)|$ elements for some H. Since G is a p-group, this is a multiple of p, and the result follows.

5. §8.4 # 5: Show that there is only one group of order 1001.

   Solution. Let $|G| = 1001 = 7 \cdot 11 \cdot 13$. We have $n_7 = 1, 11, 13, 143$ and $n_7 \equiv 1 \pmod 7$, so $n_7 = 1$. Similarly $n_{11} = 1, 7, 13, 91$ and $n_{11} \equiv 1 \pmod{11}$, so $n_{11} = 1$; and $n_{13} = 1, 7, 11, 77$ and $n_{13} \equiv 1 \pmod{13}$ so $n_{13} = 1$. Thus let $H \lhd G$, $K \lhd G$, $L \lhd G$ have order $|H| = 7$, $|K| = 11$ and $|L| = 13$. Then $H \cap K = \{1\}$ so $HK \cong H \times K \cong C_{77}$. Now $HK \cap L = \{1\}$ so $(HK)L \cong HK \times L \cong C_{1001}$. Thus $G = HKL \cong C_{1001}$ is unique up to isomorphism.