On the derivative of functions

Classroom notes for PMAT 435, Fall 2005

Abstract

This is a short outline of some topics covered in the lecture dealing with derivatives.

Lemma 0.1. *Let f with $\lim_{x\to a} f(x) = b$ be a function defined in an ϵ_2-neighborhood of a and g a function with $\lim_{x\to c} g(x) = a$. Then*

$$\lim_{x\to c} f(g(x)) = b.$$

Proof. Let $\epsilon > 0$ be given. There exists $\epsilon_1 > 0$ with $|f(x) - b| < \epsilon$ for $|x - a| < \epsilon_1$.

There exists $|\delta > 0|$ so that $|g(x) - a| < \min\{\epsilon_1, \epsilon_2\}$ for $|x - c| < \delta$.

If $|x - c| < \delta$ then $g(x)$ is in the ϵ_2-neighborhood of a and hence $f(g(x)$ is defined. Because $|g(x) - a| < \epsilon_1$ it follows that $|f(g(x)) - b| < \epsilon$.

□

Lemma 0.2. *Let f be a function which is differentiable at a. Then there exists a function $\phi(x)$ with $\lim_{x\to 0} \phi(x) = 0$ and*

$$f(a + x) - f(a) = f'(a) \cdot x + \phi(x) \cdot x. \tag{1}$$

Proof. Let

$$\phi(x) := \frac{f(a + x) - f(a)}{x} - f'(a).$$

Then $\lim_{x\to a} \phi(x) = 0$ and Equation 1 holds.

□

Theorem 0.1 (Chain Rule)**.** *Suppose that S and T are open intervals in $\mathbb{R}$ and that $f : S \to T$ and $g : T \to \mathbb{R}$ are functions. Suppose further that f is differentiable at $a \in S$ and that g is differentiable at $f(a) \in T$. Then the composite function $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'\big(f(a)\big) \cdot f'(a).$$

Proof. From Equation 1 of Lemma 0.2 we obtain:

$$g\big(f(a) + (f(a+h) - f(a))\big) - g\big(f(a)\big) = \\ g'\big(f(a)\big) \cdot (f(a+h) - f(a)) + \phi\big(f(a+h) - f(a)\big) \cdot (f(a+h) - f(a)) \tag{2}$$

with $\lim_{x \to 0} \phi(x) = 0$. Then $\lim_{h \to 0}\big(f(a+h) - f(a)\big) = 0$ because f is continuous at a and hence we get from Lemma 0.1 that

$$\lim_{h \to 0} \phi\big(f(a+h) - f(a)\big) = 0.$$

Dividing Equation 3 by h we obtain:

$$\frac{g\big(f(a+h))\big) - g\big(f(a)\big)}{h} = \\ g'\big(f(a)\big) \cdot \frac{f(a+h) - f(a)}{h} + \phi\big(f(a+h) - f(a)\big) \cdot \frac{f(a+h) - f(a)}{h}. \tag{3}$$

Hence

$$(g \circ f)'(a) = \frac{g\big(f(a+h))\big) - g\big(f(a)\big)}{h} = g'\big(f(a)\big) \cdot f'(a).$$

□

Theorem 0.2 (Darboux's Intermediate Value Theorem)**.** *Suppose that a function f is differentiable on $[a, b]$ and $f'(a) \neq f'(b)$. If k is a number between $f'(a)$ and $f'(b)$, then there exists $c \in [a, b]$ with $f'(c) = k$.*

Proof. Assume for a contradiction that there is no such $c \in [a, b]$. Then, it follows from the Mean Value Theorem, that there are no two different elements $x, y \in [a, b]$ with $\frac{f(x)-f(y)}{x-y} = k$.

Let $f'(a) < f'(b)$ and $\frac{f(b)-f(a)}{b-a} > k$. (Note that then $f'(a) < k$.) If there is an $x \in [a, b]$ with $\frac{f(x)-f(a)}{x-a} < k$ then it follows from the Intermediate Value Theorem that there is an x with $\frac{f(x)-f(a)}{x-a} = k$.

Hence, for all $x \in [a, b]$

$$f'(a) + \phi(x - a) = \frac{f(x) - f(a)}{x - a} > k$$

with $\lim_{x \to a} \phi(x - a) = 0$. This in turn implies that $f'(a) = \lim_{x \to a}\big(f'(a) - \phi(x - a)\big) \geq k$, a contradiction.

Let $f'(a) < f'(b)$ and $\frac{f(b)-f(a)}{b-a} < k$. (Note that then $f'(b) > k$.) If there is an $x \in [a, b]$ with $\frac{f(b)-f(x)}{b-x} > k$ then it follows from the Intermediate Value Theorem that there is an x with $\frac{f(b)-f(x)}{b-x} = k$.

Hence, for all $x \in [a, b]$

$$f'(b) + \phi(b - x) = \frac{f(b) - f(x)}{b - x} < k$$

with $\lim_{x \to b} \phi(b - x) = 0$. This in turn implies that $f'(b) = \lim_{x \to b}\big(f'(b) - \phi(b - x)\big) \leq k$, a contradiction.

The case $f'(a) > f'(b)$ is analogues.

□