Continuous functions, open and closed sets

Classroom notes for PMAT 435, Fall 2005

Abstract

This is a short outline concerning compact sets of reals, the Heine-Borel Theorem and applications to continuous functions.

1 Open and closed sets

Let $a, b \in \mathbb{R}$. Then $(a, b) := \{x \in \mathbb{R} : a < x \text{ and } x < b\}$. It follows that $(a, b) = \emptyset$ if and only if $b \leq a$. We call (a, b) the *open interval from a to b*. If considering an open interval (a, b) it is often tacitly assumed that $a < b$.

Also $[a, b] := \{x \in \mathbb{R} : a \leq x \text{ and } x \leq b\}$ is the *closed interval from a to b*. Note that $[a, a] = a$, that $[a, b] = \emptyset$ if and only if $b > a$ and that $[a, b] = (a, b) \cup \{a, b\}$.

Let $\mathcal{A}$ be a set of subsets of the set $\mathbb{R}$ of real numbers. That is, every element $E \in \mathcal{A}$ is a set of real numbers.

Example 1.

$$\mathcal{C} := \{(n, n+2) : n \in \omega\} = \{(0, 2), (1, 3), (2, 4), (3, 5), (4, 6), \dots\}.$$

Example 2.

$$\mathcal{D} := \{(0, n) : n \in \mathbb{N}\} := \{(0, 1), (0, 2), (0, 3), (0, 4), \dots\}$$

If $\mathcal{A}$ is a set of subsets of $\mathbb{R}$ we denote by $\bigcup \mathcal{A}$ the union of all of the elements of $\mathcal{A}$, that is:

$$\bigcup \mathcal{A} := \{x : \text{ there exists } E \in \mathcal{A} \text{ with } x \in E\}.$$

If for example $\mathcal{C}$ is as in Example 1 and $\mathcal{D}$ as in Example 2 then $\bigcup \mathcal{C} = \bigcup \mathcal{D} = (0, \infty)$, the set of non-negative reals.

The subset $S \subseteq \mathbb{R}$ is *open* if it is the union of a set of open intervals. That is, if there is a set $\mathcal{A}$ with $\bigcup \mathcal{A} = S$. It follows that every open interval and hence $\emptyset$ and $\mathbb{R}$ are open sets and that the union of open sets is open. In other words, if $\mathcal{A}$ is a set of open sets then $\bigcup \mathcal{A}$ is an open set because it will be again a union of open intervals.

It follows from Example 1 and also from Example 2 that the set $(0, \infty)$ of non negative reals is open.

Let $S \subseteq \mathbb{R}$ be an open set and $x \in S$. Because S is a union of open intervals, there exist an open interval (a, b) with $x \in (a, b) \subseteq S$. This implies that there is a real $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq S$.

Let $S \subseteq \mathbb{R}$. Then $\mathfrak{C}(S) := \{x \in \mathbb{R} : x \notin S\}$, the *complement* of S in $\mathbb{R}$.

The set $S \subseteq \mathbb{R}$ is *closed* if $\mathfrak{C}(S)$ is open. It follows that S is open if and only if $\mathfrak{C}(S)$ is closed and S is closed if and only $\mathfrak{C}(S)$ is open.

Lemma 1.1. *Let S be a subset of $\mathbb{R}$. Then S is closed if and only if every accumulation point a of S is an element of S.*

Proof. Let S be closed and a an accumulation point of S. Assume for a contradiction that $a \notin S$. Then $a \in \mathfrak{C}(S)$ which is open. Hence there is an $\epsilon > 0$ so that $(a - \epsilon, a + \epsilon) \subseteq \mathfrak{C}(S)$. That is, no element of the interval $(a - \epsilon, a + \epsilon)$ is an element of S, in contradiction to the definition of accumulation point.

Let every accumulation point of S be an element of S. Let $a \in \mathfrak{C}(S)$. Because a is not an accumulation point of S there is an $\epsilon_a > 0$ so that $(a - \epsilon_a, a + \epsilon_a) \subseteq \mathfrak{C}(S)$. It follows that $\mathfrak{C}(S)$ is the union of the open intervals $(a - \epsilon_a, a + \epsilon_a)$ over all $a \in \mathfrak{C}(S)$ and hence that $\mathfrak{C}(S)$ is open. This in turn implies that S is closed. □

Corollary 1.1. *Let S be a closed subset of $\mathbb{R}$. If $a = \sup S$ then $a \in S$ and a is the maximum of S. If S is bounded above and not empty then it has a maximum.*

If $a = \inf S$ then $a \in S$ and a is the minimum of S. If S is bounded below and not empty then it has a minimum.

2 Continuous functions

Definition 2.1 (Continuous-1). *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. The function f is* continuous-1 on D *if for every $a \in D$ and every $\epsilon > 0$ there exists a $\delta > 0$ so that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$ for all $x \in (a - \delta, a + \delta) \cap D$.*

Definition 2.2 (Continuous-2). *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. The function f is* continuous-2 on D *if for every $a \in D$ and every open subset S of $\mathbb{R}$ with $f(a) \in S$ there exists an open subset S' of $\mathbb{R}$ with $a \in S'$ so that $f(x) \in S$ for all $x \in S' \cap D$.*

Lemma 2.1. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. The function f is continuous-1 on D if and only if it is continuous-2 on D.*

Proof. Let f be continuous-1 on D. Let $a \in D$ and S an open set with $f(a) \in S$. Then there exists an $\epsilon > 0$ so that $(f(a) - \epsilon, f(a) + \epsilon) \subseteq S$. This implies that there is a $\delta > 0$ so that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon) \subseteq S$ for all $x \in (a - \delta, a + \delta) \cap D$. Let $S' := (a - \delta, a + \delta)$.

Let f be continuous-2 on D. Let $a \in D$ and let $\epsilon > 0$. Then there exists an open set T with $a \in T$ so that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$ for all $x \in T \cap D$. There exists a $\delta > 0$ so that $(a - \delta, a + \delta) \subseteq T$. It follows that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$ for all $x \in (a - \delta, a + \delta) \cap D$.

□

Let $f : D \to \mathbb{R}$ be a function and $S \subseteq \mathbb{R}$. Then $f^{-1}(S) := \{x \in D : f(x) \in S\}$. So for example if $f : [1, 3] \to \mathbb{R}$ is given by $f(x) = 2x$ for all $x \in [1, 3]$ then $f^{-1}((0, 2)) = [\frac{1}{2}, 1)$ and $f^{-1}([-2, -1]) = \emptyset$.

Definition 2.3 (Continuous-3)**.** *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. The function f is* continuous-3 on D *if for every open set S of $\mathbb{R}$ there exists an open set S' of $\mathbb{R}$ so that $S' \cap D = \{x \in D : f(x) \in S\}$.*

Note that $\{x \in D : f(x) \in S\} = f^{-1}(S)$.

Lemma 2.2. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. The function f is continuous-2 on D if and only if it is continuous-3 on D.*

Proof. Let f be continuous-3 on D. Let $a \in D$ and let S be an open subset of $\mathbb{R}$ with $f(a) \in S$. Then, because f is continuous-3, there exists an open set S' with $S' \cap D = \{x \in D : f(x) \in S\}$. From $f(a) \in S$ we get that $a \in S'$.

Let f be continuous-2 on D. Let S be an open subset of $\mathbb{R}$. If $a \in D$ with $f(a)$ in S then there exists an open subset S_a of $\mathbb{R}$ so that $f(x) \in S$ for all $x \in S_a \cap D$. Let S' be the union of the sets S_a with $f(a) \in S$.

If $x \in S' \cap D$ then there is an $a \in D$ with $x \in S_a \cap D$ and hence $f(x) \in S$. Therefore $S' \cap D \subseteq \{x \in D : f(x) \in S\}$.

If $x \in D$ with $f(x) \in S$ then $x \in S_x \cap D \subseteq S \cap D$. Hence $S' \cap D \supseteq \{x \in D : f(x) \in S\}$. □

We conclude that the three definitions of continuity are equivalent and will say that a function $f : D \to \mathbb{R}$ is *continuous on D* if it satisfies one and hence all of the definitions above.

Let S be a subset of $\mathbb{R}$. We say the subset X of S is *open in the subspace topology induced by S* if there exists an open subset T or $\mathbb{R}$ so that $X = S \cap T$.

That is, the open sets of the subspace topology on S are the intersections with S of the open sets of $\mathbb{R}$.

With this understanding we obtain from Definition 2.3 the following characterization:

Lemma 2.3. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. The function f is* continuous on D *if for every open set S of $\mathbb{R}$ the set $f^{-1}(S)$ is open in the subspace topology on S.*

3 Compact sets

Let $S \subseteq \mathbb{R}$. The set $\mathcal{A}$ of open subsets of $\mathbb{R}$ is an *open cover* of S if $S \subseteq \bigcup A$. The sets $\mathcal{C}$ and $\mathcal{D}$ of Examples 1 and 2 are open covers of the set $(0, \infty)$ and also of the set $(10, 100)$.

Let $\mathcal{A}$ be an open cover of the set $S \subseteq \mathbb{R}$. The set $\mathcal{B}$ is a *sub-cover* of S if $\mathcal{B} \subseteq \mathcal{A}$ and if $\mathcal{B}$ is a cover of S, that is $S \subseteq \bigcup \mathcal{B}$. The set $\mathcal{B}$ is a *finite sub-cover of* S of it is a sub-cover of S and finite.

The set $\mathcal{C}$ of Example 1 is an open cover of the set $(10, 100)$ and has the set $\{(n, n+2) : n \in \mathbb{N} \text{ and } 6 \leq n \leq 120\}$ as a finite sub-cover. The set $\mathcal{C}$ does not have a finite sub-cover of the set $(0, \infty)$.

Definition 3.1. *The subset $S \subseteq \mathbb{R}$ is* compact *if every open cover of S has a finite sub-cover of S.*

The set $[1, \infty)$ is not compact because it has the set $\mathcal{C}$ of Example 1 as an open cover which does not contain a finite sub-cover of $[1, \infty)$. The interval $(0, 1)$ is not compact because it has the set $\mathcal{E} := \{(\frac{1}{n}, 2) : n \in \mathbb{N}\}$ as an open cover which does not have a finite sub-cover of $(0, 1)$.

We aim to prove that every closed and bounded subset of $\mathbb{R}$ is compact. First the following.

Lemma 3.1. *Let S be a subset of $\mathbb{R}$ which is not bounded. Then S is not compact.*

Proof. Let $\mathcal{A} = \{(n, n+2) : n \in \mathbb{I}\}$. Then $\mathcal{A}$ is an open cover of S because it is an open cover of $\mathbb{R}$. If $\mathcal{B} \subseteq \mathcal{A}$ is a finite subset of $\mathcal{A}$ then $\bigcup \mathcal{A}$ is bounded. Hence $\mathcal{A}$ does not have a finite sub-cover of S. □

Lemma 3.2. *Let S be a subset of $\mathbb{R}$ which is not closed. Then S is not compact.*

Proof. According to Lemma 1.1, the set S has an accumulation point a with $a \notin S$. Let $\mathcal{A} := \{\{x \in \mathbb{R} : |a - x| > \frac{1}{n}\} : n \in \mathbb{N}\}$ then $\mathcal{A}$ is an open cover of S because it is an open cover of $\mathbb{R} \setminus \{a\}$.

Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. Then there exists an $n \in \mathbb{N}$ with $\bigcup B \subseteq \{x \in \mathbb{R} : |a - x| > \frac{1}{n}\}$. Because a is an accumulation point of S there exists an element $s \in S$ with $|a - s| < \frac{1}{n}$.

Hence $\mathcal{A}$ does not have a finite sub-cover of S. □

Lemma 3.3. *If every closed interval* $[a, b]$ *is compact, then every closed and bounded subset of* $\mathbb{R}$ *is compact.*

Proof. Let S be a closed and bounded subset of $\mathbb{R}$. Then there exists an interval $[a, b]$ with $S \subseteq [a, b]$. Let $E := (a - 1, b + 1)$ and let $F := E \setminus S$. Note that F is open because it is the intersection of the open set which is the complement of S in $\mathbb{R}$ and the open set E.

Let $\mathcal{A}$ be an open cover of S. Then $\mathcal{A} \cup \{F\}$ is an open cover of $[a, b]$. Then if $\mathcal{B}$ is a finite sub-cover of $[a, b]$ the set $\mathcal{B} \setminus \{F\}$ is a finite subset of $\mathcal{A}$ which is a cover of S.

□

Theorem 3.1 (Heine-Borel)**.** *The subset* S *of* $\mathbb{R}$ *is compact if and only if it is closed and bounded.*

Proof. According to Lemma 3.1 and Lemma 3.2 and Lemma 3.3 it remains to prove that every closed interval of the form $[a, b]$ is compact.

Assume for a contradiction that $\mathcal{A}$ is an open cover of $[a, b]$ which does not have a finite sub-cover.

If $a < x < y < b$ we say that the interval $[x, y]$ is *not finitely covered* if $\mathcal{A}$ does not have a finite subset which covers the interval $[x, y]$. Note that $[a, b]$ is not finitely covered and that if the interval $[x, y]$ is not finitely covered then the interval $[x, \frac{x+y}{2}]$ or the interval $[\frac{x+y}{2}, y]$ is not finitely covered.

It follows that there are sequences $a = \{a_i\} = a_0, a_1, a_2, a_3, \ldots$ and $b = \{b_i\} = b_0, b_1, b_2, b_3 \ldots$ with $a_i \leq a_{i+1}$ and $b_{i+1} \leq b_i$ so that the intervals $[a_i, b_i]$ are not finitely covered and so that $b_i - a_i = \frac{b-a}{2^i}$.

The sequences $\{a_i\}$ and $\{b_i\}$ are monotone and bounded and hence converge. Let $l := \lim_{n\to\infty} a_n$ and let $g := \lim_{n\to\infty} b_n$. Then $0 = \lim_{n\to\infty}(b_n - a_n) = g - l$. Hence $l = g$.

There exists an open set $E \in \mathcal{A}$ with $l \in E$ because $\mathcal{A}$ is an open cover of $[a, b]$ and $l \in [a, b]$. There exists a number $n \in \mathbb{N}$ with $(l - \frac{1}{n}, l + \frac{1}{n}) \subseteq E$. Because $\lim_{i\to\infty} a_i = l$ and $\lim_{i\to\infty} b_i = l$ there exists an index i so that $l - a_i < \frac{1}{n}$ and $b_i - l < \frac{1}{n}$.

It follows that $l - \frac{1}{n} < a_i \leq l \leq b_i < l + \frac{1}{n}$. Hence $[a_i, b_i] \subseteq E \in \mathcal{A}$ contrary to being not finitely covered. □

4 Consequences of the Heine-Borel Theorem

Theorem 4.1. *If $f : D \to \mathbb{R}$ is a continuous function on the compact set D then $\operatorname{Im}(f) := \{f(x) : x \in D\}$ is compact.*

Proof. Let $\mathcal{A}$ be an open cover of $\operatorname{Im}(f)$. Because f is continuous there exists for every set $S \in \mathcal{A}$ an open set S' with $S' \cap D = \{x \in D : f(x) \in S\}$.

The set $\mathcal{A}' := \{S' : S \in \mathcal{A}\}$ is an open cover of D, (verify), and hence has a finite sub-cover $\{S'_1, S'_2, \dots, S'_n\}$.

It follows that $\{S_1, S_2, \dots, S_n\}$ is a finite cover of $\operatorname{Im}(f)$. (Verify.)

□

Theorem 4.2. *If a function f is continuous on a closed and bounded subset S of $\mathbb{R}$ then f is bounded on S.*

Proof. Follows from Theorem 4.1 and the Heine-Borel Theorem. □

Corollary 4.1. *If a function f is continuous on a closed and bounded interval $[a, b]$ then f is bounded on $[a, b]$.*

Theorem 4.3 (Extreme Value Theorem)**.** *If f is a continuous function on a closed and bounded subset S of $\mathbb{R}$ then f attains its maximum and minimum values on S.*

Proof. Follows from Lemma 1.1and Theorem 4.1 and the Heine-Borel Theorem. □

Corollary 4.2. *If f is a continuous function on a closed interval $[a, b]$ then f attains its maximum and minimum values on $[a, b]$.*

Theorem 4.4 (Bolzano's Intermediate Value Theorem)**.** *If a function f is continuous on $[a, b]$ and if $f(a) < k < f(b)$ or if $f(a) > k > f(b)$ then there exists a real number $c \in (a, b)$ such that $f(c) = k$.*

Proof. Assume that $f(a) < k < f(b)$ and that there is no $c \in (a, b)$ with $f(c) = k$.

The sets $(-\infty, k) := \{x \in \mathbb{R} : x < k\}$ and $(k, \infty) := \{x \in \mathbb{R} : x > k\}$ are open and hence there exist, according to Definition 2.3, open sets S and T with:

$$S \cap [a, b] = \{x \in [a, b] : f(x) < k\}$$

and

$$T \cap [a, b] = \{x \in [a, b] : f(x) > k\}.$$

Note that $[a, b] \subseteq S \cup T$ and that $(S \cap [a, b]) \cap (T \cap [a, b]) = \emptyset$.

The sets S and T are open and hence unions of open intervals. Let $\mathcal{A}$ be a set of open intervals with $\bigcup \mathcal{A} = S$ and let $\mathcal{B}$ be a set of open intervals with $\bigcup \mathcal{B} = T$. Then $\mathcal{A} \cup \mathcal{B}$ is a cover of $[a, b]$ with open intervals. Because $[a, b]$ is compact there exists a finite sub-cover.

Let $\{(a_1, a'_1), (a_2, a'_2), \ldots (a_n, a'_n)\} \cup \{(b_1, b'_1), (b_2, b'_2), \ldots, (b_m, b'_m)\}$ be a cover of $[a, b]$ with $(a_i, a'_i) \in \mathcal{A}$ for all $1 \leq i \leq n$ and $(b_j, b'_j) \in \mathcal{B}$ for all $1 \leq j \leq m$.

Note if $s \in (a_i, a'_i) \cap [a, b]$ then $f(x) < k$ and if $x \in (b_j, b'_j) \cap [a, b]$ then $f(x) > k$ for all indices i, j. It follows that no interval of the form (a_i, a'_i) has a point of $[a, b]$ in common with an interval of the form (b_j, b'_j).

If $a'_i > b$ and $a_i \geq b$ then the interval (a_i, a'_i) has no point in common with $[a, b]$ and hence can be removed from the cover. If $a'_i > b$ and $a_i < b$ then $b \in (a_i, a'_i)$ in contradiction to $f(b) > k$. Hence $a'_i \leq b$. On the other hand if $a'_i \leq a$ then the interval (a_i, a'_i) has no element in common with $[a, b]$ and hence can be removed from the cover. It follows that $a'_i \in (a, b]$ for all indices i.

Let $a'_{i_0} := \max\{a'_i : 1 \leq i \leq n\}$. Then a_{i_0} is not contained in any of the intervals of the form (a_i, a'_i). Because $a'_{i_0} \in (a, b]$ there exists an index j so that $a'_{i_0} \in (b_j, b'_j)$. But this is not possible because then the intervals (a_{i_0}, a'_{i_0}) and (b_j, b'_j) would have a point of $[a, b]$ in common.

If $f(a) > k > f(b)$ the proof is analogues. □

Corollary 4.3. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then* $\mathrm{Im}(f) = \{f(x); x \in [a, b]\}$ *is a closed interval.*

Proof. Follows from Corollary 4.1 and Corollary 4.2 and the Intermediate Value Theorem. □

Definition 4.1. *Let f be a function from some subset of the reals to the reals. A number c in the domain of f with $f(c) = c$ is a* fixed point *of f.*

Corollary 4.4 (Brower's Fixed-Point Theorem)**.** *If a function $f : [a, b] \to [a, b]$ is continuous, then it has a fixed point.*

Proof. See text Theorem 4.3.10 page 165.

□

Theorem 4.5. *Let $f : D \to T$ be continuous one-to-one and onto T and let D be compact. Then $f^{-1} : T \to D$ is continuous.*

Proof. We have to prove that f^{-1} is continuous. Note that $f^{-1} : \mathrm{Im}(f) \to \mathbb{R}$ and that $\left(f^{-1}\right)^{-1} = f$. Hence, according to Definition 2.3, we have to prove that given an open subset S of $\mathbb{R}$ there exists an open subset S' of $\mathbb{R}$ so that $S' \cap \mathrm{Im}(f) = \{x \in \mathrm{Im}(f) : f^{-1}(x) \in S\}$.

Let S be an open set. Then $\mathfrak{C}(S)$ is closed and hence $\mathfrak{C}(S) \cap D$ is closed and bounded and hence compact. It follows from Theorem 4.1 that $T := \{f(x) : x \in \mathfrak{C}(S) \cap D\}$ is compact and hence closed. Hence $\mathfrak{C}(T)$ is open.

We get that $\mathfrak{C}(T) \cap \mathrm{Im}(f) = \{x \in \mathrm{Im}(f) : f^{-1}(x) \in S\}$.

□

5 Uniform Continuity

Definition 5.1. *A function $f : D \to \mathbb{R}$ is uniformly continuous on a set $E \subseteq D \subseteq \mathbb{R}$ if for any given $\epsilon > 0$ there exists $\delta > 0$ so that $|f(x) - f(z)| < \epsilon$ for all $x, z \in E$ satisfying $|x - z| < \delta$.*

If f is uniformly continuous on its domain D, we say that f is uniformly continuous.

Note that this definition of uniformly continuous can be rewritten as follows:
A function $f : D \to \mathbb{R}$ is uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ so that $f(x) \in (f(z) - \epsilon, f(z) + \epsilon)$ for all $x, z \in D$ satisfying $x \in (z - \delta, z + delta)$.

Theorem 5.1. *Let $f : D \to \mathbb{R}$ be a function with compact domain D. Then if f is continuous it is uniformly continuous.*

Proof. Let $\epsilon > 0$ be given. Then there exists, because f is continuous, for every $a \in D$ a $\delta_a > 0$ so that $f(x) \in (f(a) - \frac{1}{2}\epsilon, f(a) + \frac{1}{2}\epsilon)$ for all $x \in (a - \delta_a, a + \delta_a) \cap D$.

The set of intervals $\mathcal{A} := \{(a - \frac{1}{2}\delta_a, a + \frac{1}{2}\delta_a) : a \in D\}$ is an open cover of D. Hence, because D is compact, there exists a finite sub-cover

$$\mathcal{F} := \{(a_1 - \frac{1}{2}\delta_{a_1}, a_1 + \frac{1}{2}\delta_{a_1}), (a_2 - \frac{1}{2}\delta_{a_2}, a_2 + \frac{1}{2}\delta_{a_2}), \ldots, (a_n - \frac{1}{2}\delta_{a_n}, a_n + \frac{1}{2}\delta_{a_n})\}$$

of D.

Let $\delta := \min\{\frac{1}{2}\delta_{a_i} : 1 \leq i \leq n\}$.

Let $x, z \in D$ with $|x - z| < \delta$ and hence $x \in (z - \delta, z + \delta)$. We have to prove that $|f(x) - f(z)| < \epsilon$.

Because $\mathcal{F}$ is a cover of D there exists an a_i so that $z \in (a_i - \frac{1}{2}\delta_{a_i}, a_i + \frac{1}{2}\delta_{a_i})$ and hence

$$|z - a_i| < \frac{1}{2}\delta_{a_i} \text{ and } |f(a_i) - f(z)| < \frac{1}{2}\epsilon. \tag{1}$$

Also: $|a_i - x| = |a_i - z + z - x| \leq |a_i - z| + |z - x| < \frac{1}{2}\delta_{a_i} + \frac{1}{2}\delta \leq \delta_{a_i}$. Hence

$$|f(a_i) - f(x)| < \frac{1}{2}\epsilon. \tag{2}$$

Combining Inequalities 1 and 2 we obtain:

$$\begin{aligned} |f(z) - f(x)| =& |f(z) - f(a_i) + f(a_i)| - f(x)| \leq \\ & |f(z) - f(a_i)| + |f(a_i) - f(x)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

□