A Riemann Integral theorem

Classroom notes for PMAT 435, Fall 2005

Theorem 0.1. *Let $f \in R[a,b]$ with $m \leq f(x) \leq M$ for all $x \in [a,b]$. Let g be continuous on $[m,M]$. Then $g \circ f \in R[a,b]$.*

Proof. Let $\epsilon > 0$ be given. Let K be the difference of the maximum and minimum of g on $[m,M]$. Because f is uniformly continuous on $[m,M]$ there exists a $\delta > 0$ with $\delta < \epsilon$ so that $|g(t) - g(s)| < \frac{\epsilon}{2(b-a)}$ for $|t - s| < \delta$.

Because $f \in R[a,b]$ there exists a partition P of $[a,b]$ so that:

$$\sum_{i=1}^{n} \big(M_i(P,f) - m_i(P,f)\big)\big(x_i - x_{i-1}\big) < \frac{\delta^2}{2K}. \tag{1}$$

Let S be the set of all indices i with $M_i(P,f) - m_i(P,f) \leq \delta$ and T the set of all indices i with $M_i(P,f) - m_i(P,f) > \delta$. Note that if $i \in S$ then

$$M_i(P, g \circ f) - m_i(P, g \circ f) \leq \frac{\epsilon}{2(b-a)}. \tag{2}$$

and if $i \in T$ then

$$\sum_{i \in T} \delta(x_i - x_{i-1}) < \sum_{i \in T} \big(M_i(P,f) - m_i(P,f)\big)\big(x_i - x_{i-1}\big) < \frac{\delta^2}{2K} \tag{3}$$

and hence that

$$\sum_{i \in T} (x_i - x_{i-1}) < \frac{\delta}{2K} < \frac{\epsilon}{2K}. \tag{4}$$

Let $M_i(P, g \circ f) := M_i^*$ and $m_i(P, g \circ f) := m_i^*$. Then

$$\sum_{i=1}^{n} \big(M_i^* - m_i^*\big)\big(x_i - x_{i-1}\big) =$$

$$\sum_{i \in S} \big(M_i^* - m_i^*\big)\big(x_i - x_{i-1}\big) + \sum_{i \in T} \big(M_i^* - m_i^*\big)\big(x_i - x_{i-1}\big) \leq$$

$$\sum_{i \in S} \frac{\epsilon}{2(b-a)}(x_i - x_{i-1}) + \sum_{i \in T} K(x_i - x_{i-1}) < \frac{\epsilon}{2(b-a)}(b-a) + K\frac{\epsilon}{2K} = \epsilon.$$

□