The derivative of e^x

Classroom notes for PMAT 435, Fall 2005

Theorem 0.1. *The derivative of the function* $f := e^x$ *at* $a \in \mathbb{R}$ *is* e^a.

Proof. Remember that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and that $\lim_{x\to 0} e^x = 1$.

Then, for $x \neq 0$:

$$\frac{e^x - 1}{x} = \frac{1}{x}(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots) = 1 + \frac{1}{2}x + \frac{1}{3}\frac{x^2}{2!} + \frac{1}{4}\frac{x^3}{4!} + \dots$$

Hence:

$$1 \leq \frac{e^x - 1}{x} \leq e^x$$

which implies that $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

Then:

$$\lim_{h\to 0} \frac{e^{a+h} - e^a}{h} = \lim_{h\to 0} \frac{e^a e^h - e^a}{h} = \lim_{h\to 0} \frac{e^a(e^h - 1)}{h} = e^a \lim_{h\to 0} \frac{e^h - 1}{h} = e^a \cdot 1 = e^a.$$

□