PMAT 435 (Spring 2008) Midterm - Solution and Comments

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1. Mark each statement True or False. Justify each answer.

   (a) (2 marks) If $m = \inf S$ and $m' < m$, then m' is a lower bound for S.
   [Solution. TRUE. Since $m = \inf S$, m is a lower bound for S. So, for all $s \in S$, we have $m \leq s$. Since $m' < m$, we have $m' < m \leq s$ for all s in S. This means that m' is a lower bound for S.
   Comments: The most common problem is the following. Saying that because m is the greatest of the lower bounds, any number less than it must be a lower bound. The problem is that it is not clear to me whether you want to claim it is the fact that m is a lower bound that m' must be a lower bound or it is the fact that no lower bound can be greater than m that m' must be a lower bound. The first reasoning is correct, but the second reasoning is incorrect. The fact that no numbers that have a particular property (here, this property is 'being a lower bound') can be greater than m does not necessarily mean that all numbers smaller than m must have that property. Your justification was not considered to be clear enough if you did not single out in your argument the 'relevant' fact that m is a lower bound.]

   (b) (2 marks) The union of any collection of closed sets is closed.
   [Solution. FALSE. Consider the collection $\left\{\left[\frac{1}{n}, 2\right] : n \in \mathbf{N}\right\}$. For each $n \in \mathbf{N}$, $\left[\frac{1}{n}, 2\right]$ is closed. However, the union of the collection is $\cup_{n=1}^{\infty}\left[\frac{1}{n}, 2\right] = (0, 2]$, which is not closed.
   Comments: Common problems include: (1) Appealing to the theorem that says the union of finitely many closed sets is closed and arguing that because of this the statement is true only if we area dealing with a finite collection. This reasoning is faulty because the most one can say is that that theorem does not give us the guarantee that the union of an infinite collection of closed sets is closed. It by itself does not say that the union of an infinite collection of closed sets is not necessarily closed. Rather, it simply does not address the scenario concerning infinite collections. So, the theorem is insufficient (or one might even say 'irrelevant') for showing that the statement is false. Similarly, some of you claim that the statement is true ONLY IF the collection is finite. This is ambiguous. It could mean either (a) It is not the case that the union of an infinite collection of closed sets is necessarily closed, or (b) it is necessarily not the case that the union of an infinite collection of closed sets is closed. (a) is true, and is what you need to justify; (b) is false. It could happen that the union of infinitely many closed sets turn out to be closed. An example would be $\mathbf{N} = \cup_{n=1}^{\infty}\{n\}$. (2) Giving a 'counterexample' that does not work, i.e., what was proposed as a counterexample was not a counterexample. For instance, the union of the collection $\{[-n, n] : n \in \mathbf{N}\}$ is the set $\mathbf{R}$ of all real numbers. Note, however, that the set of all real numbers IS closed. A similar example is this: The union of the collection $\{[0, n] : n \in \mathbf{N}\}$ is the set $[0, \infty)$. Note, however, that this set IS closed. (3) Understanding 'not closed' to mean 'open'. Always remember that all these four possibilities exist: Some sets are both open and closed, some sets are open but not closed, some sets are closed but not open, and some sets are neither open nor closed. If you are to decide whether or not a set is closed, it will not do if you just study whether or not the set is open. Knowing whether or not a set is open does not directly tell you whether or not that set is closed (unless you

also use other pre-established facts).]

(c) (2 marks) Given sequences (s_n) and (a_n), if for some $s \in \mathbf{R}$, $k > 0$ and $m \in \mathbf{N}$ we have $|s_n - s| \leq k\,|a_n|$ for all $n > m$, then $\lim s_n = s$.
[Solution. FALSE. Consider the counterexample $s_n = (-1)^n$, $s = 0$, $k = 1$, $a_n = 1$, $m = 1$. Then we have, for all $n > m$, $|s_n - s| = |(-1)^n - 0| = 1 = k\,|a_n|$. However, $\lim s_n \neq s$.
Comments: Common problems include: (1) Appealing to theorem 16.8 and saying that because here we do not require $a_n \to 0$, the statement is not true. This reasoning is faulty for the same reason as discussed in Comment (1) for part (b) above. (2) Not giving a full counterexample. Some of you just say $s_n = -1$. The problem with this is that it does not supply everything needed for a counterexample. A counterexample should contain a sequence (s_n), a number s, another sequence (a_n), a positive number k, and a positive integer m. All of these, taken together, form the counterexample if $|s_n - s| \leq k\,|a_n|$ for all $n > m$ but $\lim s_n \neq s$. It turns out that if you choose $s_n = -1$ for all n, then it will not constitute a counterexample if s is chosen to be -1.]

2. (5 marks) For this question, you may choose to do EITHER part (a) OR part (b). If S is a compact subset of $\mathbf{R}$ and T is a closed subset of S, then T is compact.

(a) Prove this using the definition of compactness.
[Solution. I did this one in class.]

(b) Prove this using the Heine-Borel theorem.
[Solution. Since S is compact, it follows from the Heine-Borel theorem that S is bounded. So, there is an upper bound, say b, for S and a lower bound, say a, for S. Thus, for all s in S, we have $a \leq s \leq b$. Not, if t is any element of T, t is also an element of S, since $T \subseteq S$. But then $a \leq t \leq b$. This proves that a and b are also lower and upper bounds, respectively, for T. Thus, T is bounded. Now, since T is closed, it follows from the Heine-Borel theorem that T is compact.
Comment: There is only one imperfection I want to mention. Some of you say that because S is compact and T is a subset of S, T must be bounded. This jumps a step. The thing is it is not clear whether you want to say that T is bounded because S is closed or because S is bounded. A clear argument will avoid lumping together relevant points with irrelevant points. The fact that S is closed is irrelevant, and you should single out the boundedness of S to argue for the boundedness of T.]

3. (5 marks) Prove this theorem: The set $\mathbf{R}$ of real numbers is uncountable.
[Solution. Not much to say.]

4. (4 marks) Use the definition of the limit of a convergent sequence (i.e., For every $\varepsilon > 0$, there exists some N such that ...) to show that $\lim (2n^3 + 1) / (n^3 - 4n + 5) = 2$.
[Solution. I did this one in class.
Comments: Common problems include: (1) Not being able to find N from the target inequality $|s_n - s| < \varepsilon$. (2) Arguing in the wrong direction. Some of you wrote what was in effect this argument: $|s_n - s| < \varepsilon \Rightarrow s_n - s < \varepsilon \Rightarrow ...$ THe problme with this is that although it is true that if $|s_n - s| < \varepsilon$, then $s_n - s < \varepsilon$, it is not enough to find the condition $n > N$ from the inequality $s_n - s < \varepsilon$. If you (simplfy and) solve the inequality $s_n - s < \varepsilon$ to obtain some inequality of the form $n >$ some N, it at best means that you have found N such that $n > N$ implies $s_n - s < \varepsilon$. It is still not proved

that $n > N$ would imply $|s_n - s| < \varepsilon$. Because $|s_n - s| < \varepsilon$ might not be true just because $s_n - s < \varepsilon$ is true.]
